Key Facts

PASS GCE

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A-LEVEL PURE MATHEMATICS

A.K.Beard B.Sc.

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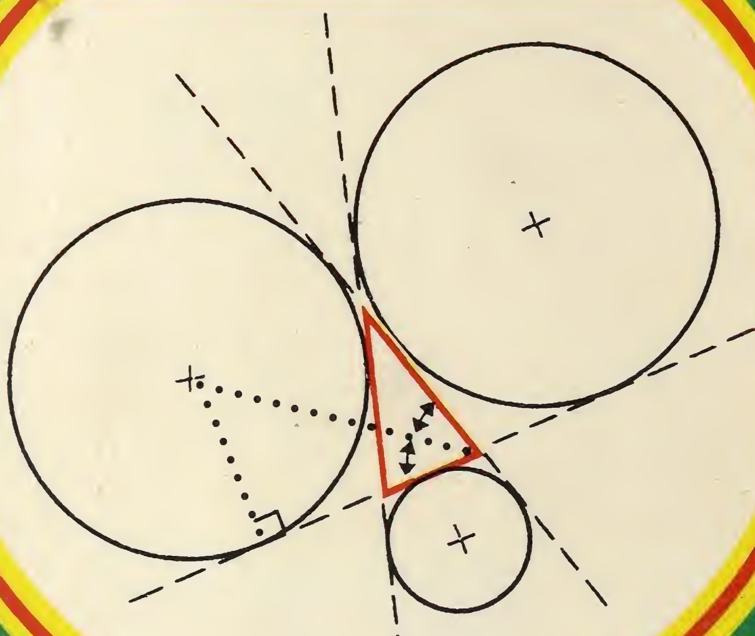

PURE MATHEMATICS

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Algebra

ARITHMETIC PROGRESSION

The sequence of numbers 2, 5, 8, 11, 14, ... is such that any number, or **term,** can be found by adding 3 to the preceding term. We say two consecutive terms have a **common difference** of 3. Such a sequence is called an **Arithmetic Progression** or A.P.
The terms of the A.P. can be written as follows

$2, 2 + (1 \times 3), 2 + (2 \times 3), 2 + (3 \times 3), 2 + (4 \times 3), \ldots$

and so, if **a** is the **first term** and **d** is the common difference, the
$\mathbf{n^{th}}$ **term** $= \mathbf{a + (n - 1)d}$ (1)
If an A.P. has a finite number of terms, say n, then we can find the sum S_n of the A.P. by the formula

$$\mathbf{S_n = \tfrac{1}{2}n[2a + (n - 1)d]} \qquad (2)$$

The **last term l** of this A.P. is the nth term, so that $l = a + (n - 1)d$

$$\therefore\ \mathbf{S_n = \tfrac{1}{2}n(a + l)} \qquad (3)$$

An important result to remember is the sum of the first n natural numbers

$1 + 2 + 3 + \cdots + n = \tfrac{1}{2}n(n + 1)$

The **Arithmetic Mean** of two numbers a, b, is the number which when placed between them, makes three terms of an A.P. Hence

Arithmetic Mean $= \tfrac{1}{2}(a + b)$

A formula for the **sum** of a series can be used to deduce a formula for the $\mathbf{n^{th}}$ **term.** This is the sum of the first **n** terms minus the sum of the first $\mathbf{n - 1}$ terms, i.e.

$S_n - S_{n-1} = T_n.$

Thus, if the sum of n terms of a series is $5n^2 - 14n$,

$$T_n = 5n^2 - 14n - 5(n - 1)^2 + 14(n - 1)$$
$$= 10n - 19 = -9 + 10(n - 1),$$

and the series is an A.P. with first term -9 and c.d. 10.
Given the sum, common difference and first term of an A.P. formula (2) gives a **quadratic equation** to find n. When $d > 0$ and $S_n > 0$, one of the roots will be negative, but if $d < 0$ and $S_n > 0$, there will be **two positive roots** which, if integral, will be possible values of n.

The numbers between say 100 and 500 which are divisible by 7 form an A.P. of first term 105 and last term 497, so that $n = 57$, and $S = 57(105 + 497) \div 2 = 17{,}157$.

GEOMETRIC PROGRESSION

The sequence of numbers

2, 6, 18, 54 ...

is such that any term can be found by multiplying the preceding term by 3. In other words, two consecutive terms have a **common ratio** of 3. Such a sequence is called a **Geometric Progression** or G.P. The terms of the G.P. can be written as follows

$2, 2 \times (3)^1, 2 \times (3)^2, 2 \times (3)^3, \ldots$

and so, if a is the first term and r the common ratio, the **n^{th} term is ar^{n-1}**

The sum $\mathbf{S}_n$ of the first n terms of a G.P. is given by the formula

$$\mathbf{S}_n = \frac{\mathbf{a(r^n - 1)}}{\mathbf{r - 1}} \quad \text{or} \quad \frac{\mathbf{a(1 - r^n)}}{\mathbf{1 - r}}$$

If $-1 < r < 1$, $r^n \to 0$ as $n \to \infty$. Hence, if a G.P. has common ratio r such that $-1 < r < 1$, the sum to infinity S_∞ exists and is given by

$$\mathbf{S}_\infty = \frac{\mathbf{a}}{\mathbf{1 - r}}.$$

Two important results to remember are

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1 - x}$$

$$1 - x + x^2 - x^3 + \cdots = \frac{1}{1 + x}$$

provided $-1 < x < 1$.
The expression $-1 < x < 1$ is often written as $|\mathbf{x}| < \mathbf{1}$, read as 'mod x'. Mod x means the numerical value of x.
Particular terms of a series are found approximately by logs, if n is large, while the problem of finding the number of terms to give a certain sum needs care, especially if $r < 1$.

For example, to find the number of terms of the series: $16 + 12 + 9 + 6\frac{3}{4} \ldots$ needed to exceed 63·9,

we have $$63{\cdot}9 = 16 \times \frac{1 - (0{\cdot}75)^n}{1 - 0{\cdot}75}$$

i.e. $$63{\cdot}9 = 64 - 64 \times (0{\cdot}75)^n$$

$$\therefore \quad 0{\cdot}75^n = 0{\cdot}1 \div 64.$$

Taking logs gives $n \times \bar{1}{\cdot}8751 = -1 - 1{\cdot}8062$

or $$-0{\cdot}1249n = -2{\cdot}8062,$$

so that $$n = \frac{2{\cdot}8062}{0{\cdot}1249} = 22{\cdot}4$$

It follows that 23 terms will be needed to give $S > 63{\cdot}9$.

BINOMIAL THEOREM, POSITIVE INTEGER

If n is a positive integer then, whatever the value of x, the following expansion holds.

$$(1 + x)^n = 1 + \frac{n}{1}(x) + \frac{n(n-1)}{1.2}(x^2) + \frac{n(n-1)(n-2)}{1.2.3}(x^3) + \cdots$$

This is called the **Binomial Expansion** of $(1 + x)^n$. Note carefully the pattern of the expansion; the powers of x are steadily increasing and the coefficient of x^r has r factors in the numerator and r factors in the denominator. Further, the factors in the numerator start at n and come down in 1's and those of the denominator start at 1 and go up in 1's.

The **coefficient of x^r** is sometimes written ${}^nC_r$ or $\binom{n}{r}$

Note that, since n is a positive integer, the numerator eventually produces a zero factor. The expansion therefore has a finite number $(n + 1)$ of terms. Remember also that the coefficients read the same from right to left as they do from left to right. The theorem can be extended to $(a + x)^n$ where a takes any value.

$$(a + x)^n = a^n\left(1 + \frac{x}{a}\right)^n = a^n\left\{1 + \frac{n}{1}\left(\frac{x}{a}\right) + \frac{n(n-1)}{1.2}\left(\frac{x}{a}\right)^2 + \cdots\right\}$$

$$= a^n + \frac{n}{1}.xa^{n-1} + \frac{n(n-1)}{1.2}x^2a^{n-2} + \cdots$$

Use of theorem
The theorem can be used to find a positive integer power of say N, by writing $N = a + x$ where $\frac{x}{a}$ is small

e.g. $(10{\cdot}1)^5 = \left(10 + \frac{1}{10}\right)^5 = 10^5\left(1 + \frac{1}{100}\right)^5$

$$= 10^5 + (5 \times 10^3) + 10^2 + 1 + \frac{5}{10^3} + \frac{1}{10^5}$$

$$= 105101{\cdot}00501$$

Another application often met with is in finding, from first principles, the **differential coefficient** of say x^3.

If $y = x^3$.

$$y + \delta y = (x + \delta x)^3 = x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3$$

so that $\dfrac{\delta y}{\delta x} = 3x^2 + 3x(\delta x) + (\delta x)^2$

BINOMIAL THEOREM, INDEX NOT A POSITIVE INTEGER

The form of the expansion remains the same

$$(1 + x)^q = 1 + \frac{q}{1}(x) + \frac{q(q-1)}{1.2}(x^2) + \cdots$$

Now, however, the expansion is only true if $-1 < x < 1$.

ex. $(1 + x)^{1/3} = 1 + \dfrac{\frac{1}{3}}{1}x + \dfrac{\frac{1}{3}(\frac{1}{3} - 1)}{1.2}x^2 + \dfrac{\frac{1}{3}(\frac{1}{3} - 1)(\frac{1}{3} - 2)}{1.2.3}x^3 +$

$$= 1 + \tfrac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 + \cdots \text{provided } -1 < x < 1.$$

ex. $(1 - x)^{-1} = 1 + \dfrac{(-1)}{1}(-x) + \dfrac{(-1)(-2)}{1.2}(-x)^2 +$

$$\frac{(-1)(-2)(-3)}{1.2.3}(-x)^3 + \cdots$$

$$= 1 + x + x^2 + x^3 + \cdots \text{provided } -1 < x < 1,$$

a result seen before under G.P.

Note that the series cannot terminate.

As before, $(a + x)^q = a^q\left(1 + \frac{x}{a}\right)^q$ and the expansion can be applied provided that $-1 < \frac{x}{a} < 1$.

ex. $\frac{1}{4} = 4^{-1} = (3 + 1)^{-1} = 3^{-1}(1 + \frac{1}{3})^{-1} = \frac{1}{3}(1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots)$
is **true,** since $-1 < \frac{1}{3} < 1$.
But $\frac{1}{4} = (1 + 3)^{-1} = (1 - 3 + 9 - 27 + \cdots)$
is **false,** since $-1 < 3 < 1$ does not hold.

Use of the expansion

The expansion is very useful in finding **approximate values of** N^q Write $N = a + x$ with the conditions that

(i) $-1 < \frac{x}{a} < 1$ and (ii) a^q is readily known.

ex. $\frac{1}{8{\cdot}24} = (8 + 0{\cdot}24)^{-1} = 8^{-1}(1 + 0{\cdot}03)^{-1} \simeq \frac{1}{8}(1 - 0{\cdot}03) = 0{\cdot}12125$

ex. Use the expansion of $(1 - 0{\cdot}02)^{1/2}$ to find $\sqrt{2}$ to 5 decimal places.

We have $1 - 0{\cdot}02 = \frac{98}{100}$, so that $\sqrt{1 - 0{\cdot}02} = \frac{7}{10}\sqrt{2}$

hence $\quad \frac{7}{10}\sqrt{2} = 1 - 0{\cdot}01 - 0{\cdot}00005 - 0{\cdot}0000005$

$\therefore \quad \sqrt{2} = 9{\cdot}899495 \div 7 = 1{\cdot}41421(4)$.

(see also example on page 24)

EXPONENTIAL SERIES

The power series

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^r}{r!} + \cdots$$

is an infinite series which converges to a limit for every finite value of x. The sum of the series is denoted by $\mathbf{e^x}$. The value of the **number e** may be calculated to any required degree of accuracy by setting $x = 1$ in the above expression. Correct to 6 significant figures, $e = 2{\cdot}71828$.

Properties of e^x

(i) e^x increases as x increases for all values of x
(ii) $e^x \to \infty$ as $x \to \infty$
(iii) $e^x \to 0$ as $x \to -\infty$. (iv) $e^0 = 1$.

Notice that e^x is **always positive.**
The number e is important as the base of natural or Napierian logarithms.

LOGARITHMIC SERIES

The series

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad (-1 < x \le 1)$$

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \qquad (-1 \le x < 1)$$

are known as the **logarithmic series.** Take particular note that

(i) the second series may be obtained by putting $x = -x$ in the first series
(ii) the denominators of each term are single numbers unlike the factorials which appear in the exponential series
(iii) the ranges of values of x for which the series are valid are most important.

Subtracting the series we obtain, after simplification,

$$\tfrac{1}{2}\log_e\left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots \qquad (-1 < x < 1)$$

This series, because it converges more rapidly than either of those above, is useful in numerical working.

Example Find the value of $\log_e 5$ to two decimal places.

We need $\dfrac{1+x}{1-x} = 5$. i.e. $x = \frac{2}{3}$

$$\text{then, } \log_e 5 = \log_e\left(\frac{1+\frac{2}{3}}{1-\frac{2}{3}}\right) = 2\left(\frac{2}{3} + \frac{1}{3}\left(\frac{2}{3}\right)^3 + \frac{1}{5}\left(\frac{2}{3}\right)^5 + \frac{1}{7}\left(\frac{2}{3}\right)^7 + \frac{1}{9}\left(\frac{2}{3}\right)^9 + \cdots\right)$$

$$= 2\,(0{\cdot}667 + 0{\cdot}099 + 0{\cdot}026 + 0{\cdot}008 + 0{\cdot}003 + \cdots)$$

$$= 2 \times 0{\cdot}803 \simeq 1{\cdot}61$$

QUADRATIC FUNCTIONS AND EQUATIONS

The roots of the quadratic equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(i) If $(b^2 - 4ac) > 0$, then $\sqrt{b^2 - 4ac}$ exists as a real number and the two roots are **real** and different.

(ii) If $b^2 - 4ac = 0$, then the roots are **both** $-\frac{b}{2a}$

(iii) If $b^2 - 4ac < 0$, then $\sqrt{b^2 - 4ac}$ does not exist as a real number and the two roots are **unreal** or **imaginary** or **complex.**

$(b^2 - 4ac)$ is called the **discriminant** as, clearly, it discriminates between the different types of roots.
The graph of $y = ax^2 + bx + c$ is a parabola and the solution of $0 = ax^2 + bx + c$ is given by the intersection of the parabola with the x-axis. The above cases are equivalent to
(i) cutting the x-axis twice;
(ii) touching the x-axis;
(iii) not cutting the x-axis at all.
Using the discriminant, we can deduce some of the properties of **quadratic functions** of the type

$$\frac{ax^2 + bx + c}{Ax^2 + Bx + C}$$

Example Consider the possible values of $y = \frac{x^2 + 3x + 2}{x^2 + 2x - 3}$

Rearranging as a quadratic in x we have,

$x^2(y - 1) + x(2y - 3) - (3y + 2) = 0$

For real x, the discriminant must be ≥ 0.

i.e. $(2y - 3)^2 + 4(y - 1)(3y + 2) \geq 0$

i.e. $16y^2 - 16y + 1 \geq 0$
Suppose α and β are the roots of $16y^2 - 16y + 1 = 0$.

Then $\quad 16y^2 - 16y + 1 = (y - \alpha)(y - \beta)$

hence $\quad (y - \alpha)(y - \beta) \geq 0$

Thus y cannot lie between α and β, that is between $\frac{2 \pm \sqrt{3}}{4}$.

ROOTS OF EQUATIONS

The equation $(x-\alpha)(x-\beta)=0$, has roots $x=\alpha$ and $x=\beta$.
Conversely, if an equation has roots α and β, it can be written in the form $(x-\alpha)(x-\beta)=0$.
Hence, if the equation $ax^2+bx+c=0$ has roots α and β then

$$x^2+\frac{b}{a}x+\frac{c}{a}\equiv(x-\alpha)(x-\beta)$$

$$=x^2-(\alpha+\beta)x+\alpha\beta$$

Comparing coefficients

$$\alpha+\beta=-\frac{b}{a} \qquad \text{(sum of roots)}$$

$$\alpha\beta=\frac{c}{a} \qquad \text{(product of roots)}$$

Using these two statements, some functions of α and β can be evaluated.

Example 1 If $x^2+4x+2=0$ has roots α and β, then

$\alpha+\beta=-4$ and $\alpha\beta=2$

(i) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=12$

(ii) $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=-40$

(iii) $\alpha-\beta=\pm\sqrt{\alpha^2-2\alpha\beta+\beta^2}=\pm\sqrt{(\alpha+\beta)^2-4\alpha\beta}$
$=\pm\sqrt{8}$

Example 2 To form from $x^2+2x+3=0$ the equation with roots $\alpha+3\beta$, $3\alpha+\beta$, we have, if S, P are the sum and product of these roots

$S=(\alpha+3\beta)+(3\alpha+\beta)=4(\alpha+\beta)=-8$

$P=(\alpha+3\beta)(3\alpha+\beta)=3\alpha^2+10\alpha\beta+3\beta^2=3(\alpha+\beta)^2+4\alpha\beta=24$

The required equation is $x^2-Sx+P=0$ or $x^2+8x+24=0$.

A useful technique to remember is that α and β must satisfy the given equation.

i.e. $\alpha^2 + 4\alpha + 2 = 0$ and $\beta^2 + 4\beta + 2 = 0$

Hence $\alpha^2 + \beta^2 = -(4\alpha + 2) - (4\beta + 2) = -4(\alpha + \beta) - 4 = 12$

Similarly $\alpha^3 + \beta^3 = (-4\alpha - 2)\,\alpha + (-4\beta - 2)\,\beta$

$$= -4(\alpha^2 + \beta^2) - 2(\alpha + \beta) = -48 + 8 = -40$$

The process can be repeated to find $\alpha^n + \beta^n$.
Similar results to those shown above exist for cubic, quartic and higher degree equations.
If the roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$ are α, β, and γ, then we can write

$$ax^3 + bx^2 + cx + d \equiv (x - \alpha)(x - \beta)(x - \gamma)$$

hence $\alpha + \beta + \gamma = -\dfrac{b}{a}$ (sum of roots)

$\alpha\beta + \beta\gamma + \gamma\alpha = \dfrac{c}{a}$ (sum of products of roots in pairs)

$\alpha\beta\gamma = -\dfrac{d}{a}$ (product of roots)

1. If $x^3 + bx^2 + cx + d = 0$ has roots α, β, γ, express in terms of b, c, d, the expression $\alpha^3 + \beta^3 + \gamma^3$.

We have $\alpha + \beta + \gamma = -b$, $\alpha\beta + \beta\gamma + \gamma\alpha = c$, $\alpha\beta\gamma = -d$. Also,

$$\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha)$$

$$\therefore\ \alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma + (\alpha + \beta + \gamma)\{(\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)\}$$

$$= -\{(3d + b(b^2 - 3c)\}$$

2. If the roots of the equation $ax^3 + bx^2 + cx + d = 0$ are in A.P. prove that $b^3 = \frac{1}{2}a(9bc - 27ad)$
Let the roots be $\alpha + h$, α, $\alpha - h$.

Then $\alpha + h + \alpha + \alpha - h = 3\alpha = -\dfrac{b}{a}$ (1)

$$\alpha(\alpha + h) + \alpha(\alpha - h) + (\alpha + h)(\alpha - h) = \frac{c}{a}$$

that is $3\alpha^2 - h^2 = \dfrac{c}{a}$ (2)

$(\alpha + h)\alpha(\alpha - h) = \alpha(\alpha^2 - h^2) = -\dfrac{d}{a}$ (3)

Substituting from 1 and 3 into 2 gives

$$\frac{2b^2}{9a^2} + \frac{3d}{b} = \frac{c}{a}$$

$$2b^3 + 27a^2d = 9abc$$

$$b^3 = \tfrac{1}{2}a(9bc - 27ad).$$

REMAINDER THEOREM

If a polynomial $P(x) \equiv a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ is divided by $(x - \alpha)$, the **remainder,** which must be of **smaller degree** than $(x - \alpha)$, is a **constant.**

The **remainder theorem** states that this constant is $P(\alpha)$. This can be seen by writing

$$P(x) = (x - \alpha)\, Q\,(x) + \text{Remainder}$$

and writing $x = \alpha$. We have $\mathbf{P(\alpha) = 0 + remainder.}$

In particular, if there is **no** remainder, then $(x - \alpha)$ is a **factor** of $P(x)$.

If $P(x)$ is divided by both $(x - \alpha)$ and $(x - \beta)$, the remainder will be of degree less than $(x - \alpha)\,(x - \beta)$. It will therefore have the form $Ax + B$.

Writing $P(x) \equiv (x - \alpha)\,(\alpha - \beta)\,Q(x) + (Ax + B)$

and substituting $x = \alpha$ and $x = \beta$ we have

$$P(\alpha) = A\alpha + B$$

$$P(\beta) = A\beta + B$$

from which A and B can be found.

Example 1 Solve the equation $4x^3 + 16x^2 - 23x - 15 = 0$.

Putting $x = 1$, the remainder is $4 + 16 - 23 - 15 \therefore x = 1$ is **not** a root.

Putting $x = -\frac{1}{2}$ the remainder is $4(-\frac{1}{8}) + 16(\frac{1}{4}) - 23\,(-\frac{1}{2}) - 15 = 0$.

Therefore $(2x + 1)$ is a factor and we have by long division or inspection

$$(2x + 1)(2x^2 + 7x - 15) = 0$$

or $(2x + 1)(2x - 3)(x + 5) = 0$.

Hence $x = -\frac{1}{2}$, $1\frac{1}{2}$ or -5.

Example 2 Prove that $(a + 2b + 3c)$ is a factor of $a^3 + 8b^3 + 27c^3 - 18abc$. If $a + 2b + 3c$ is a factor, the result of substituting $-(\mathbf{2b + 3c})$ for **a** should give a zero remainder.

We have: Remainder $= (-2b - 3c)^3 + 8b^3 + 27c^3 - 18\,(-2b - 3c)bc$
$= -8b^3 - 36^2c - 54bc^2 - 27c^3 + 8b^3 + 27c^3 + 36b^2c + 54bc^2 = 0.$

Hence $(a + 2b + 3c)$ **is a factor** of the expression.

PERMUTATIONS AND COMBINATIONS

Most of the results that will be required in this section of the syllabus depend upon the principle that;

if a task may be performed in **p** ways and a second independent task in **q** ways, then the two tasks can be performed together in **pq** ways.

A **permutation** of a group of different objects is a rearrangement of the group in some other order.
e.g. 1, 3, 2 is a permutation of 1, 2, 3.
The **number of permutations** of r different objects can be found in the following way.
Object No. 1 can be chosen in **r ways.**
Object No. 2 can be chosen in **(r − 1) ways** since **one object has gone.**
Hence Object No. 1 together with Object No. 2 can be chosen in $r(r-1)$ different ways.
Continuing thus, the number of permutations of the r objects is

$$\mathbf{r(r-1)(r-2)(r-3)\ldots 3.2.1}$$

or $1 \times 2 \times 3 \times \ldots \times (r-3) \times (r-2) \times (r-1) \times r$.

This repeated multiplication is called **factorial r** and written $r!$ or $\underline{r|}$.
The convention is used that $0! = 1$.
The number of permutations of r objects taken from n different objects is written ${}^n\mathbf{P}_r$.
Arguing as above, it can easily be seen that

$${}^n\mathbf{P}_r = n \times (n-1) \times (n-2) \times \ldots \text{for } r \text{ factors}$$

$$= \frac{\mathbf{n!}}{\mathbf{(n-r)!}}$$

A group of r things with **no regard to the order of arrangement** is called a **combination.**
The number of combinations of r objects taken from n different objects is written

$${}^n\mathbf{C}_r \quad \text{or} \quad \binom{n}{r}$$

Each member of ${}^nP_r$ consists of r objects which can be arranged amongst themselves in $r!$ ways.

$$\therefore {}^n\mathbf{C}_r = \frac{{}^n\mathbf{P}_r}{\mathbf{r!}} = \frac{\mathbf{n!}}{\mathbf{(n-r)!\,r!}}$$

which is the general form for the Binomial coefficients for positive integer n.

There are two important results concerning combinations.

(i) ${}^n\mathbf{C}_r = {}^n\mathbf{C}_{n-r}$. This result is obvious if we realise that as we choose r things we are simultaneously rejecting $n - r$. So for every choice of r things there is an equal choice of $n - r$. Try calculating ${}^{10}C_4$ and ${}^{10}C_6$.

(ii) ${}^n\mathbf{C}_r + {}^n\mathbf{C}_{r+1} = {}^{n+1}\mathbf{C}_{r+1}$. This is used in proving both the Binomial theorem and Leibnitz' theorem.
Let a be one of a group of $n + 1$ things, from which $r + 1$ are to be selected. This can be done in ${}^{n+1}C_{r+1}$ ways.
Now if a is to be definitely included in the selection it can be made in ${}^nC_r$ ways, and if a is to be definitely excluded from the selection it can be made in ${}^nC_{r+1}$ ways.
Hence ${}^{n+1}C_{r+1} = {}^nC_r + {}^nC_{r+1}$.

Example 1 Find the number of different forecasts which can be given on the results of 6 football matches.
Each match has 3 different possible forecasts.
$\therefore$ The total number of forecasts is $3^6 = \mathbf{729}$.

Example 2 Find the number of different combinations of three letters which can be taken from the word ALGEBRA.
There are 5 different letters and 2 A's. The possibilities are

0 A's + 3 from 5 which gives ${}^5C_3$ ways. (1)
1 A + 2 from 5 which gives ${}^5C_2$ ways. (2)
2 A's + 1 from 5 which gives ${}^5C_1$ ways. (3)

Total number of **combinations** is ${}^5C_3 + {}^5C_2 + {}^5C_1$.
The number of **arrangements** of **all** the letters from 'Algebra' would be 7! if all the letters were different, but is $7! \div 2!$, i.e. 2520 **because of the repeated a.**
In general, if a group of **n** letters consists of **p** A's, **q** B's, **r** C's, and the other letters are all different, the number of **permutations** will be

$$\frac{\mathbf{n!}}{\mathbf{p!\,q!\,r!}}$$

The number of arrangements of **some** letters in these cases must be found **after** the selections have been made. In the problem above the number of **arrangements** for selections (1) and (2) is 3! while for group (3) it is $3! \div 2$. Hence the number of arrangements is $10 \times 3! + 10 \times 3! + 5 \times 3 = 135$.

Some or All

The number of ways in which a person can choose some or all from a collection of different things is best dealt with on the basis of selecting or rejecting each individual article. Each thing may be taken or left, we have two choices for each. The total number of selections is thus $2 \times 2 \times 2 \ldots$ to n terms, i.e. 2^n. However this includes the case where all n articles have been rejected and so the total number of different selections is

$$2^n - 1.$$

A similar problem arises with the shopkeeper who possesses 3 100g weights, 1 50g weight and 4 10g weights. How many different weights can be made up using them? 0, 1, 2 or 3 of the 100g weights may be used, 0 or 1 50g and 0, 1, 2, 3 or 4 of the 10g's. The correct number of 39 arises as $4 \times 2 \times 5 - 1$.

Circular Permutations

In the case of 10 people seated at a round table, one person, A, can take his place first and the other nine arrange themselves in 9! ways relative to A. We are not concerned with the positions in relation to a particular point on the table. In the case of 10 keys on a key-ring, the fact the ring can be turned over and viewed from both sides means that the number of arrangements is halved, there being no difference between clockwise and anti-clockwise arrangements.
General results for both cases are $(n - 1)!$ and $\frac{1}{2}(n - 1)!$ $(n > 2)$.

Separation

To find the number of arrangements of 6 men and 4 women in a row so that no two women sit together, arrange the men first in 6! ways, leaving gaps between them. Including the positions beyond the end men there are 7 gaps, which the four women can partly occupy in ${}^7P_4$ ways.

Hence we have $\dfrac{6!7!}{3!}$ arrangements.

PROBABILITY

When a die is thrown there are six possible outcomes, 1, 2, 3, 4, 5 or 6. These are equally likely to happen assuming that we have an unbiased or 'fair' die. Therefore the probability of any one number arising is 1/6. We define the probability (p) of an event happening as

$$\frac{\textbf{the number of outcomes favourable to the event}}{\textbf{the total number of possible outcomes}}$$

assuming that all outcomes are **equally likely.**

Note

(i) The probability of an event lies between 0 and 1. The probability of an event which is certain to happen is 1, that of one which cannot possibly happen is 0.

(ii) If the probability of an event happening is p then the probability of it not happening is $1 - p = q$

(iii) Care must be shown in using probability to forecast the likely result of some event. In many games there are three possible results, a win for either side or a draw. It would be wrong to conclude that the probability of a draw is $\frac{1}{3}$. This is true only if the results are equally likely; much more is needed to decide this.

Combined Events

If two **independent** events have probabilities of p_1 and p_2 respectively, then the probability of the events happening together is $p_1 \times p_2$. The condition that the event must be truly independent is most important.

Odds

A common use of probability is in stating the odds against or for an event. If the probability of an event is 1/11, then in a sufficiently large number of trials we would expect 1/11 to be successful and 10/11 to be failures. We say that the odds are **10 to 1 against** the event happening. If the probability is 3/4, the odds are **3 to 1 on** the event happening. If failure and success are equally likely we say that the odds are **evens.**

Example 1 A bag contains 5 red counters and 7 blue. If two counters are taken at random from the bag, find the probability that they are **both blue.**

Number of ways of getting 2 blues $= {}^7C_2 = 21$

Number of ways of picking 2 counters $= {}^{12}C_2 = 66$

$$\therefore p = \frac{21}{66} = \frac{7}{22}$$

To find the probability that **at least one** is blue, it is quicker to find q, i.e., the probability that **neither is blue.**

Number of ways of getting 2 reds $= {}^5C_2 = 10.$

$$\therefore q = \frac{10}{66} = \frac{5}{33}, \quad \text{and} \quad p = 1 - q = \frac{28}{33}$$

Example 2 A letter is drawn from the alphabet and is then replaced.

(i) What is the chance that the letter drawn will be a vowel? There are five possible favourable outcomes (a, e, i, o, u) from the twenty-six letters. ∴ Chance (probability) = 5/26.

(ii) What is the chance that in two consecutive draws the letter drawn will be the same?

We are not concerned with the first draw. The chance that any particular letter will come out in the second draw is 1/26. If the first letter drawn was B then we need a B to be drawn in the second draw. The chance is clearly 1/26.

(iii) What is the chance that in 100 consecutive draws the A will not be drawn?

The chance of an A being drawn is 1/26 in any one draw.
∴ the chance of it not being drawn is $1 - 1/26 = 25/26$.
∴ the chance of it not being drawn on 100 consecutive occasions is

$$\left(\frac{25}{26}\right)^{100} = 0{\cdot}0195 \qquad \text{(by logs.)}$$

A distinction must be made in cases when the first of a number of happenings affects later ones. **Note carefully the difference here.** The probability of getting a total of 11 from **two throws** of a six-sided die is 1/18, since the two successful occurrences are 6 + 5 and 5 + 6, and there are 6 ways of the die falling on the first occasion, and 6 again on the second.
However, if we have six cards numbered 1 to 6, the probability

of **drawing 2 cards** giving a total of 11 is 1/15.
This follows either by saying there is one favourable selection out of ${}^6C_2$ selections, or two favourable arrangements ($6 + 5$ or $5 + 6$) out of ${}^6P_2$ or 30 permutations.
In the case of the die the **probability of a total less than 11** is 1 − (probability of 12) − (probability of 11), which is

$$1 - \frac{1}{36} - \frac{1}{18} = \frac{11}{12}.$$

INDUCTION

Induction is a method of proving the truth of a formula for **positive integers.**

Note (i) the formula **must** be known or guessed;
(ii) the proof is for positive integers only.

The method consists in
(i) assuming the formula is **true** for a general integer **n** and then **proving** it is also true for the integer $(n + 1)$;
(ii) **proving** the formula is true for $n = 1$.

This concludes the proof because, if the formula is true for $n = 1$, then by (i) above it has been proved true for $n = 2$ and hence for $n = 3$ and so on, i.e. it is **true for all integers n.**

Example 1 Prove that

$$S_n = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n}{6}(2n + 1)(n + 1).$$

(i) Assume S_n is true.

$$\therefore\ S_{n+1} = S_n + (n + 1)^2 = \frac{n}{6}(2n + 1)(n + 1) + (n + 1)^2$$

$$= \frac{n + 1}{6}(2n^2 + 7n + 6)$$

$$= \frac{(n + 1)}{6}\{2(n + 1) + 1\}\{(n + 1) + 1\}.$$

and so the formula is true for $(n + 1)$.
(ii) If $n = 1$, $S_n = 1^2 = 1$ and the formula gives $\frac{1}{6}(3)(2) = 1$, which is true.

This concludes the proof.

Example 2 Prove that if **n** is a positive integer,

$3^{2n+2} - 8n + 55$ is divisible by 64.

Denote $3^{2n+2} - 8n + 55$ by u_n

then if (1) $u_{n+1} - u_n$ is divisible by 64 for all n

and (2) u_1 is divisible by 64, all u_r's will be divisible by 64.

Now $u_{n+1} = 3^{2(n+1)+2} - 8(n+1) + 55$

$= 3^{2n+4} - 8n + 47$

$\therefore u_{n+1} - u_n = 3^{2n+4} - 3^{2n+2} - 8$

$= 3^{2n+2}(3^2 - 1) - 8$

$= 8(3^{2n+2} - 1).$

But $a^{2k} - 1$ has factors **(a − 1), (a + 1).** Hence $u_{n+1} - u_n$ has factors 8, 2, 4 or 64. Also $u_1 = 81 - 8 + 55 = 128 = 2.64$.

PARTIAL FRACTIONS

The function $\dfrac{x+3}{x^2-1}$ can be written as $\dfrac{2}{x-1} - \dfrac{1}{x+1}$

We say that $\dfrac{x+3}{x^2-1}$ has been expressed in **partial fractions.**

Any Rational Algebraic Fraction can be expressed in Partial Fractions. To do this

(i) Make sure the numerator is of **smaller degree** than the denominator. If it is not, **divide first**

e.g. $\dfrac{x^3}{x^2-1}$ **must** be written as $x + \dfrac{x}{x^2-1}$ and the function $\dfrac{x}{x^2-1}$ written in Partial Fractions.

(ii) **Factorise the denominator.** For every factor $(x - \alpha_1)$ there is a fraction $\dfrac{A}{x-\alpha_1}$; for every factor $(x^2 + \alpha_1 x + \beta_1)$ there is a fraction $\dfrac{A_1 x + B_1}{x^2 + \alpha_1 x + \beta_1}$ and so on. The **numerator** is always of **degree one less** than the **denominator.**

If a factor is repeated, say $(x - \alpha_1)^2$, then there are **two** associated

fractions $\dfrac{A}{x - \alpha_1}$ and $\dfrac{B}{(x - \alpha_1)^2}$.

If $(x - \alpha_1)^3$, there are three associated fractions

$$\frac{A}{x - \alpha_1}, \quad \frac{B}{(x - \alpha_1)^2} \quad \text{and} \quad \frac{C}{(x - \alpha_1)^3}$$

and so on.

To find the constants A, B ... etc., first put all the fractions on a Common Denominator and then **compare the two numerators.**

$$\text{e.g.} \ \frac{x + 3}{x^2 - 1} = \frac{x + 3}{(x - 1)(x + 1)} \equiv \frac{A}{x - 1} + \frac{B}{x + 1} = \frac{A(x + 1) + B(x - 1)}{x^2 - 1}$$

$\therefore x + 3 \equiv A(x + 1) + B(x - 1)$.

At this point it is well to remind ourselves that the above expression is **an identity,** true for all values of x. Identities also have the important property that coefficients of like powers of x are equal. These properties give us two methods for getting the values of A and B.

(i) **Compare coefficients.**

Coefficient of x; $1 = A + B$.

Constant term; $3 = A - B$

(ii) **Substitute** suitable values of x in both sides.

take $x = 1$; $4 = 2A$

take $x = -1$; $2 = -2B$.

Do you see how 'suitable' values for x were chosen? The choice of $x = 1$ was made so that the term containing B was reduced to zero and a simple equation in A was left. Similarly the choice of $x = -1$ eliminated the term containing A. Any other pair of values would have led to two simultaneous equations in A and B. It would not be wrong to use that method but it would be inefficient. Sometimes it cannot be avoided. Note that a combination of **both** methods can be used, i.e. **suitable substitutions** and the comparison of **convenient coefficients.** The finding of Partial Fractions is very important in certain **expansions** and **integrals** as the Partial Fractions are much easier to handle.

Example Express in partial fractions $\dfrac{x^2 + 5x + 10}{x^3 - 8}$

We have $\dfrac{x^2 + 5x + 10}{x^3 - 8} \equiv \dfrac{A}{x - 2} + \dfrac{Bx + C}{x^2 + 2x + 4}$ (see page 14)

$$\therefore \quad x^2 + 5x + 10 \equiv A(x^2 + 2x + 4) + (x - 2)(Bx + C) \qquad (1)$$

Put $x = 2$, $4 + 10 + 10 = A(4 + 4 + 4) + 0$

$$\therefore \qquad 2 = A \qquad (2)$$

Substitute 2 for A in (1) and rewrite as

$-x^2 + x + 2 \equiv (x - 2)(Bx + C)$

Unless an error has been made, L.H.S. must have a factor $(x - 2)$

Here $(x - 2)(-x - 1) \equiv (x - 2)(Bx + C)$

so that $\qquad -x - 1 \equiv Bx + C$

and the partial fractions are $\dfrac{2}{x - 2} - \dfrac{x + 1}{x^2 + 2x + 4}$.

An **alternative** method, after obtaining **A**, is to say:
Compare coefficients of x^2 in (1) $1 = A + B \quad \therefore B = -1$
Compare number terms in (1) $10 = 4A - 2C \quad \therefore C = -1$.
Another **useful device** occurs with **repeated** factors.

Example Express $\dfrac{5x^2 + 3x - 2}{(x - 2)^3}$ in partial fractions.

Put $y = x - 2$, i.e. $x = y + 2$, and the fraction becomes

$$\frac{5(y + 2)^2 + 3(y + 2) - 2}{y^3}$$

$$= \frac{5y^2 + 20y + 20 + 3y + 6 - 2}{y^3}$$

$$= \frac{5}{y} + \frac{23}{y^2} + \frac{24}{y^3}$$

or $$= \frac{5}{x - 2} + \frac{23}{(x - 2)^2} + \frac{24}{(x - 2)^3}$$

A particularly important pair of partial fractions for **integration purposes** is given now.

$$\frac{1}{x^2 - a^2} \equiv \frac{A}{x-a} + \frac{B}{x+a}$$

Here $1 \equiv A(x + a) + B(x - a)$

and we find $A = \dfrac{1}{2a}$, $B = \dfrac{-1}{2a}$,

so that $\dfrac{1}{x^2 - a^2} \equiv \dfrac{1}{2a(x-a)} - \dfrac{1}{2a(x+a)}$

Example Express $\dfrac{1+x}{(1-x)(1+x^2)}$ in partial fractions and hence obtain the series of ascending powers of x for the expression. If terms above x^6 may be neglected show that the expression reduces to $(1 + x)^2(1 + x^4)$.

Let $\dfrac{1+x}{(1-x)(1+x^2)} \equiv \dfrac{A}{1-x} + \dfrac{Bx + C}{1+x^2}$

Whence $1 + x \equiv A(1 + x^2) + (1 - x)(Bx + C)$

and we find $A = 1$, $B = 1$, $C = 0$.

The partial fractions are $\dfrac{1}{1-x} + \dfrac{x}{1+x^2}$

$= (1 - x)^{-1} + x(1 + x^2)^{-1}$

Using the expansions for $(1 \pm x)^{-1}$ (see p. 8)

$$\begin{aligned}\frac{1+x}{(1-x)(1+x^2)} &= (1 + x + x^2 + x^3 + x^4 + x^5 + \cdots) \\ &\quad + x(1 - x^2 + x^4 - \cdots) \\ &= 1 + 2x + x^2 + x^4 + 2x^5 + x^6 + x^8 + 2x^9 + \cdots\end{aligned}$$

neglecting terms of higher degree than x^6 leads to

$$\begin{aligned}(1 + 2x + x^2) + (x^4 + 2x^5 + x^6) &= (1 + x)^2 + x^4(1 + x)^2 \\ &= (1 + x)^2(1 + x^4)\end{aligned}$$

INDICES

The **laws** governing indices are as follows:

$\mathbf{a^p \times a^q = a^{p+q}}$

In particular; taking $q = 0$,
$a^p \times a^o = a^p$, whence $\mathbf{a^o = 1}$

and, taking $q = -p$,

$a^p \times a^{-p} = a^{p-p} = a^o = 1$. Whence $\mathbf{a^{-p} = \dfrac{1}{a^p}}$.

$\dfrac{\mathbf{a^P}}{\mathbf{a^Q}} = a^p \times a^{-q}$ whence $\dfrac{\mathbf{a^P}}{\mathbf{a^Q}} = \mathbf{a^{P-Q}}$

Writing $p = q = \frac{1}{2}$, $a^{1/2} \times a^{1/2} = a^1 = a$.

Hence $\mathbf{a}^{1/2}$ is another way of writing $\sqrt{\mathbf{a}}$.
Similarly $a^{1/3}$ is another way of writing $\sqrt[3]{a}$.
Finally, $\mathbf{(a^P)^Q = a^{PQ}}$.
Hence, for example, $a^{2/3} = a^{2 \times 1/3} = (a^2)^{1/3}$ or $(a^{1/3})^2$.
In dealing with fractional negative indices, it is best to deal with the negative index first. Thus

$$16^{-3/4} = \frac{1}{16^{3/4}} = \frac{1}{(\sqrt[4]{16})^3} = \frac{1}{2^3} = \frac{1}{8}.$$

LOGARITHMS

Given any two **positive** numbers a and y, it is possible to find x such that

$a^x = y$.

The number x is called the **logarithm** of **y** to the **base a.** We write

$\mathbf{x = \log_a y}$.

In other words, $\log_a y$ is that power to which the base a must be raised to give y.
e.g. Since $100 = 10^2$, it follows that $\log_{10} 100 = 2$, and since $a^0 = 1$, it follows that $\log_a 1 = 0$.
Since then a **logarithm is an index,** the properties of indices can be used to prove

(i) $\mathbf{\log (p \times q) = \log p + \log q}$;

(ii) $\log \frac{p}{q} = \log p - \log q$;

(iii) $\log p^q = q \log p$.

In Mathematical processes, such as calculus, the base that arises naturally is **e.** Such **logs to base e** are called **natural** or **napierian** logs. In **ordinary calculations,** because the powers of 10 are so easy to handle, **the base chosen is 10.** Such logs are called **common** logs.

It is sometimes necessary to change from one base to another. To do this, use

$$\log_a p \times \log_p q = \log_a q$$

A useful "help" to the memory is to delete the word log, and regard the other numbers as fractions

$$\log_a p \times \log_p q = \log_a q \qquad \frac{p}{a} \times \frac{q}{p} = \frac{q}{a}$$

$$\text{e.g. } \log_a p \times \log_p a = 1 \qquad \frac{p}{a} \times \frac{a}{p} = 1.$$

$$\therefore \log_a p = \frac{1}{\log_p a}.$$

In particular it should be noted that

$$\log_e a \times \log_{10} e = \log_{10} a \qquad \frac{a}{e} \times \frac{e}{10} = \frac{a}{10}.$$

i.e. $\mathbf{0{\cdot}4343 \log_e a = \log_{10} a.}$

Since $2^3 = 8$, we have $\log_2 8 = 3$.
Similarly $\log_8 2 = \frac{1}{3}$, and $\log_3 81 = 4$.
Use may be made of properties of logarithms to build up the beginnings of a table of logarithms to 7 decimal places.

$$\begin{aligned}
\text{We have } \log 2 &= 0{\cdot}3010300 \\
\log 3 &= 0{\cdot}4771213 \\
\text{Hence } \log 18 &= \log 2 + 2 \log 3. \\
&= 1{\cdot}2552726 \\
\text{and } \log 75 &= \log 300 - 2 \log 2 \\
&= 1{\cdot}8750613.
\end{aligned}$$

When **drawing graphs of log x** for values of $\mathbf{x < 1}$, logs must be written as **completely negative numbers.**

Another case where this technique was necessary occurred on page 7, when finding an **index associated with a number** < 1.
For logs to base 10, however, it is nearly always simpler to use the 'bar' notation with which the student is familiar.
In the case of **natural** logarithms, i.e., logs to base **e,** tables of which are often headed **ln x,** the "bar" notation would be useless.

We write $\log \frac{1}{3} = \log 1 - \log 3 = -\log 3.$
$= -\mathbf{1{\cdot}0986}$

Similarly $\log 30 = \log 3 + \log 10 = \mathbf{1{\cdot}0986} + 2{\cdot}3026$
$= \mathbf{3{\cdot}4012}$

and $\log 0{\cdot}03 = \log 3 - \log 100$
$= 1{\cdot}0986 - 4{\cdot}6052$
$= -\ 3{\cdot}5066$

Example 1 Solve the equations $3^x = 27^{y+1}$, $2^{x-9} = 4^y$.

Equ. 1 gives $3^x = (3^3)^{y+1} = 3^{3y+3}$

hence $\quad x = 3y + 3 \quad (3)$

Equ. 2 gives $2^{x-9} = (2^2)^y = 2^{2y}$

$x - 9 = 2y \quad (4)$

Solving (3) and (4) simultaneously gives $x = 21$ and $y = 6$.

Example 2 Solve the equations $\log_y x - 9 \log_x y = 0$, $\log_{10} x + \log_{10} y = 1$.

Expressing equ. 1 in terms of $\log_{10}$ by the change of base rule,

gives $\dfrac{\log_{10} x}{\log_{10} y} - \dfrac{9 \log_{10} y}{\log_{10} x} = 0$

hence $\quad (\log_{10} x)^2 = 9(\log_{10} y)^2$

$\therefore \quad \log_{10} x = 3 \log_{10} y \quad$ (A)

Substitute in the second equation,

$3 \log_{10} y + \log_{10} y = 1$

$\therefore \quad \log_{10} y = \frac{1}{4} = 0{\cdot}2500 \qquad y = 1.778$

hence $\quad \log_{10} x = \frac{3}{4} = 0{\cdot}7500 \qquad x = 5{\cdot}623$

Note: Why did we only consider the positive square root in line A?

INEQUALITIES

Inequalities can be handled in a similar way to equations. Thus,

if $p > q$, then $p + r > q + r$
if $p > q$, then $pr > qr$ when $r > 0$ but $pr < qr$ when $r < 0$
if $p > q$, then $1/p < 1/q$
if $p > q > 0$, then $p^n > q^n$ when $n > 0$ but $p^n < q^n$ when $n < 0$.

Note that, when multiplying or dividing by a negative number, the inequality sign is reversed. So if $p > q$ and $p > 0$ and $q < 0$

Then $\frac{1}{p} > \frac{1}{q}$.

Examples

$$
\begin{array}{llll}
 & & 3 > 2 & \\
\therefore & 10 = & 7 + 3 > 7 + 2 & = 9 \\
 & -4 = & 3 - 7 > 2 - 7 & = -5 \\
 & 21 = & 3 \times 7 > 2 \times 7 & = 14
\end{array}
$$

But $-21 = 3 \times -7 < 2 \times -7 = -14$
and $-3/7 = 3 \div -7 < 2 \div -7 = -2/7$

Remember that a **square of a real number cannot be negative.** This simple fact can lead to useful inequalities.
For instance, if a and b are real,

$(a - b)^2 \geq 0$

$\mathbf{a^2 + b^2 \geq 2ab}$

In particular, if a and b are positive, write $a = \sqrt{x}$ and $b = \sqrt{y}$

then $\frac{x + y}{2} \geq \sqrt{xy}$

i.e. for two real numbers, **Arithmetic Mean ≥ Geometric Mean.** When associated with **completing the square,** inequalities make a powerful tool.
The **minimum** value of $3x^2 - 7x + 1$ can be found by writing it as

$3(x^2 - \frac{7}{3}x) + 1 = 3[x^2 - \frac{7}{3}x + (\frac{7}{6})^2] + 1 - 3(\frac{7}{6})^2 = 3(x - \frac{7}{6})^2 - \frac{37}{12}$

Hence since $(x - \frac{7}{6})^2 \geq 0$ for all x, the minimum value of the expression is $-\frac{37}{12} = -3\frac{1}{12}$, and it occurs when $x = 1\frac{1}{6}$. On page 6 the modulus of a number was introduced. The following inequalities are of interest.

(i) $|\mathbf{x}| > 0$ for all $x \neq 0$, $x = 0$ if $|x| = 0$.

(ii) $|\mathbf{xy}| = |\mathbf{x}| \cdot |\mathbf{y}|$

(iii) $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$

If $x = 6$ and $y = -2$ then $|6 - 2| = 4 < |6| + |-2| = 6 + 2 = 8$.

Example 1 Prove that for real a and b, $a^3 + b^3 \geq a^2b + ab^2$.

$a^3 + b^3 - a^2b - ab^2 = (a - b)(a^2 - b^2)$

$(a - b)$ and $(a^2 - b^2)$ are both positive when $a > b$ and both negative when $a < b$. Zero when $a = b$.

$\therefore\ a^3 + b^3 \geq a^2b + ab^2$.

Example 2 Prove that for real a, b, c and d, $(a^2 + b^2)(c^2 + d^2) \geq (ac + bd)^2$. (This is an example of the Cauchy inequality).

$$(a^2 + b^2)(c^2 + d^2) = (a^2c^2 + b^2d^2) + (a^2d^2 + b^2c^2)$$
$$= (ac + bd)^2 + (ad + bc)^2 - 4abcd$$

The result is proved if we can show that $(ad + bc)^2 \geq 4abcd$.
We have, from the previous page, that $A.M. \geq G.M.$

i.e. $ad + bc \geq 2\sqrt{ad \cdot bc}$

$(ad + bc)^2 \geq 4abcd$

We are successful. Note that the equality holds if $a/c = b/d$. The same result can be used quite frequently, for instance, as in proving that

$(b + c)(c + a)(a + b) \geq 8abc$.

$$(b + c)(c + a)(a + b) \geq 2\sqrt{bc} \,.\, 2\sqrt{ca} \,.\, 2\sqrt{ab}$$
$$= 8abc$$

Trigonometry

CIRCULAR MEASURE

Given a circle centre O, radius **r**; take 2 points A and B on the circumference such that the **arc length** $AB = \mathbf{r}$. Then we define the angle AOB as equal to **1 radian** and write 1^c.
Changing the value of r produces a similar figure, hence the angle AOB is independent of r.
Since 2π arcs of length r can be taken on the circumference it follows that $\mathbf{360^\circ = 2\pi}$ **radians**

$$\text{or } 1^c = \frac{360^\circ}{2\pi} \simeq 57{\cdot}3^\circ.$$

Conversion from degrees to radians can be done by using the relationship $360^\circ = 2\pi^c$ or by using tables. Remember, when using tables, that **radians and degrees are directly proportional** to each other. Hence 100° can be taken as $2 \times 50^\circ$ or $90^\circ + 10^\circ$.
If the points A and B are such that angle $AOB = \theta^c$, then arc length $AB = r\theta$.

$$\text{The area of the sector } AOB = \frac{\theta}{2\pi} \times \pi r^2 = \tfrac{1}{2}r^2\theta.$$

For small values of x, measured in **radians.**

$\sin x \simeq x$
$\tan x \simeq x$
$\cos x \simeq 1 - \frac{1}{2}x^2$

Circular measure has **three** important uses.
1. The fact that for **small** angles, $x \simeq \sin x \simeq \tan x$, enables us to find good **approximations** to the size of distant objects.
An object 1 cm. high, 1 metre away subtends an angle whose sin, tangent and radian measure are all approximately 0·01, giving an angle of 34′, or 35′. The same object at a distance of 1 km. would subtend an angle of 0·00001 radian or about 2″.
2. The formula $\frac{1}{2}r^2\theta$ for **sector area** is much simpler to use than the clumsy $\dfrac{\theta}{360}.\pi r^2$, which is used if the angle is given in degrees.

3. The main value of circular measure is however only found when we start dealing with **calculus.** Differentiation and integration of ratios of angles measured in degrees would be so cumbersome as to prove almost impracticable.

TRIGONOMETRICAL FUNCTIONS

Conventionally, an angle is traced out by starting from the positive direction of the X-axis.
The **anti-clockwise** direction is considered **positive,** and the clockwise direction negative.
It follows that angles of the form $\mathbf{360n}^\circ + \theta$, where n is any positive or negative integer, are in the **same position.**
The value of a trigonometrical function sin θ, etc. for θ **in the range** 0° to 90°, can be found from the tables. For θ **not in this range,** apply the following **rules.**

(i) Find the **magnitude** by **repeatedly subtracting 180°** from the angle *or* the angle from 180°, until an angle is obtained in the range 0° to 90°

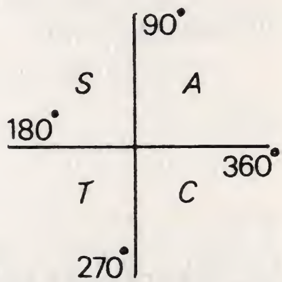
(ii) Find the **sign** by using the **CAST diagram.** This diagram is formed by writing **C**(*os*) in the **south-east quadrant** of angles (i.e. 270° to 360°) and then writing the letters **A**(*ll*), **S**(*in*), **T**(*an*) in the other quadrants going anti-clockwise.

The letters indicate **when the trig. function is positive.**
These rules follow from the **extended definition** of trigonometric ratios. If P is the point (x, y) and $OP = \mathbf{r}$, we have

$$\textbf{sin } \angle XOP = \frac{\textbf{y coordinate}}{\textbf{r}}$$

$$\textbf{cos } \angle XOP = \frac{\textbf{x coordinate}}{\textbf{r}}$$

$$\textbf{tan } \angle XOP = \frac{\textbf{y coordinate}}{\textbf{x coordinate}}$$



e.g. **sin 950°**

(i) **magnitude** sin (950°–5.180°) = sin 50°

(ii) **sign.** 950° lies in the $T(an)$ quadrant and so the sine is negative.
∴ sin 950° = –sin 50°

e.g. **Sin 1000°**

(i) **magnitude** sin (1000° – 5.180°) = sin 100° = sin (180° – 100°) = sin 80°

(ii) **sign.** 1000° lies in the $C(os)$ quadrant and so the sine is negative.

$\therefore \sin 1000^\circ = -\sin 80^\circ$

From the extended definitions given above it is still true to say that

$$\frac{\sin\theta}{\cos\theta} = \tan\theta; \qquad \frac{\cos\theta}{\sin\theta} = \cot\theta.$$

ADDITION FORMULAE AND THEIR CONSEQUENTS

The following formulae called the **Addition Formulae** should be remembered.

(i) $\sin(A+B) = \sin A\cos B + \cos A\sin B.$ (1)

(ii) $\sin(A-B) = \sin A\cos B - \cos A\sin B.$ (2)

(iii) $\cos(A+B) = \cos A\cos B - \sin A\sin B.$ (3)

(iv) $\cos(A-B) = \cos A\cos B + \sin A\sin B.$ (4)

(v) $\tan(A+B) = \dfrac{\tan A + \tan B}{1-\tan A\tan B}$ (5)

(vi) $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A\tan B}$ (6)

Special and important identities can be obtained by

(i) writing $A = 0$;

$$\sin(-B) = -\sin B \qquad (7)$$
$$\cos(-B) = \cos B \qquad (8)$$
$$\tan(-B) = -\tan B \qquad (9)$$

(ii) writing $B = A$;

$$\sin 2A = 2\sin A\cos A \qquad (10)$$
$$\cos 2A = \cos^2 A - \sin^2 A \qquad (11)$$
$$1 = \cos^2 A + \sin^2 A \qquad (12)$$
$$\tan 2A = \frac{2\tan A}{1-\tan^2 A} \qquad (13)$$

Using $1 = \cos^2 A + \sin^2 A$, other forms of cos 2A are obtained.

$\cos 2A = 2\cos^2 A - 1$ or $\cos^2 A = \frac{1}{2}(1+\cos 2A)$ (14)

$\cos 2A = 1 - 2\sin^2 A$ or $\sin^2 A = \frac{1}{2}(1-\cos 2A).$ (15)

Again, dividing $1 = \cos^2 A + \sin^2 A$ by $\cos^2 A$

$\sec^2 A = 1 + \tan^2 A$ (16)

and dividing by $\sin^2 A$

$\operatorname{cosec}^2 A = 1 + \cot^2 A.$ (17)

Extensions can be made thus:

$$\begin{aligned}\mathbf{sin\ 3A} &= \sin(2A + A) \\ &= \sin 2A \cos A + \cos 2A \sin A \\ &= 2 \sin A \cos A \,.\, \cos A + (1 - 2\sin^2 A)\sin A \\ &= \mathbf{3\ sin\ A - 4\ sin^3\ A.}\end{aligned} \tag{18}$$

Similarly $\mathbf{cos\ 3A = 4\ cos^3\ A - 3\ cos\ A.}$ (19)

Two very important results for sin 2A and cos 2A follow:

$$\sin 2A = 2 \sin A \cos A = \frac{2 \sin A \cos A}{\cos^2 A + \sin^2 A}$$

Now divide both numerator and denominator by $\cos^2 A$.

Then $$\mathbf{sin\ 2A = \frac{2\ tan\ A}{1 + tan^2\ A}} \tag{20}$$

Similarly $$\mathbf{cos\ 2A = \frac{1 - tan^2\ A}{1 + tan^2\ A}} \tag{21}$$

FACTOR FORMULAE

The following formulae called the factor formulae should also be remembered.

$$\mathbf{sin\ A + sin\ B = 2\ sin\ \frac{A + B}{2}\ cos\ \frac{A - B}{2}} \tag{22}$$

$$\mathbf{sin\ A - sin\ B = 2\ cos\ \frac{A + B}{2}\ sin\ \frac{A - B}{2}} \tag{23}$$

$$\mathbf{cos\ A + cos\ B = 2\ cos\ \frac{A + B}{2}\ cos\ \frac{A - B}{2}} \tag{24}$$

$$\mathbf{cos\ A - cos\ B = -2\ sin\ \frac{A + B}{2}\ sin\ \frac{A - B}{2}} \tag{25}$$

Note carefully the **negative** sign in the last formula.

It is sometimes necessary to use these formulae in reverse. Then

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B) \tag{26}$$

$$2 \cos A \sin B = \sin(A + B) - \sin(A - B) \tag{27}$$

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B) \tag{28}$$

$$-2 \sin A \sin B = \cos(A + B) - \cos(A - B) \tag{29}$$

Note that the 2's in the denominators have disappeared.

Example 1 Prove that if $A + B + C = 180°$. then $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$

$\cos^2 A + \cos^2 B + \cos^2 C$
$= \frac{1}{2}(1 + \cos 2A) + \frac{1}{2}(1 + \cos 2B) + \cos^2 C$ using (14)
$= 1 + \frac{1}{2}\cos 2A + \frac{1}{2}\cos 2B + \cos^2 C$
$= 1 + \cos(A + B)\cos(A - B) + \cos^2 C$ using (24)
$= 1 + \cos(A + B)\{\cos(A - B) + \cos(A + B)\}$ $-\cos(A + B) = \cos C$
$= 1 + \cos(A + B)2\cos A \cos B$ using (24)
$= 1 - 2\cos A \cos B \cos C.$ $-\cos(A + B) = \cos C$

Example 2 In a triangle ABC prove that

$$\cot A = \frac{c - a\cos B}{a \sin B}$$

Recall that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ i.e. the sine rule.

Then we may substitute $c = 2R\sin C$ and $a = 2R\sin A$

$$\therefore \frac{c - a\cos B}{a\sin B} = \frac{2R\sin C - 2R\sin A\cos B}{2R\sin A\sin B}$$

$$= \frac{\sin(A + B) - \sin A\cos B}{\sin A\sin B} \qquad \sin(A + B) = \sin C$$

$$= \frac{\sin A\cos B + \cos A\sin B - \sin A\cos B}{\sin A\sin B} \qquad \text{using (1)}$$

$$= \frac{\cos A\sin B}{\sin A\sin B} = \frac{\cos A}{\sin A} = \cot A$$

INVERSE TRIGONOMETRIC FUNCTIONS

The statement $\sin x = a$ can also be written, $\mathbf{x = \sin^{-1} a}$.
$\text{Sin}^{-1} a$ is called the inverse sine. It is an angle, namely the angle whose sine is a.

Note very carefully that $\sin^{-1} a$ is **not** $\frac{1}{\sin a}$.

$\sin^{-1}a$ is frequently written as arc sin a to avoid this confusion.

The other inverse functions are similarly defined.
Any inverse function can be changed into another by the use of Pythagoras Theorem in a **suitable right-angled triangle.**
e.g. If $\mathbf{\sin^{-1} a = x}$, then x can be regarded as the base angle in a right-angled triangle of height $\mathbf{a}$ and hypotenuse 1. Hence the base is $\sqrt{1 - a^2}$ and hence
$\mathbf{x = \cos^{-1}\sqrt{1 - a^2}}$, etc.

In the same way, we can see that
$\sin^{-1}(\frac{3}{5}) = \cos^{-1}(\frac{4}{5}) = \tan^{-1}(\frac{3}{4})$,
all being equal to the smallest angle in a (3,4,5) triangle.

Example 1 Find the value of x if

$$\tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{5} + \tan^{-1}\frac{1}{x} = \frac{\pi}{4}.$$

The difficulty with most trig. problems is in choosing how to arrange the data and to select an appropriate formula from the many available. In this question we are presented with angles and tangents. This suggests using (5) and (6). So rearrange to give 2 pairs of angles. Then

$$\tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{5} = \frac{\pi}{4} - \tan^{-1}\frac{1}{x}$$

$$\tan(\tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{5}) = \tan\left(\frac{\pi}{4} - \tan^{-1}\frac{1}{x}\right)$$

$$\frac{\frac{1}{2} + \frac{1}{5}}{1 - \frac{1}{2}\cdot\frac{1}{5}} = \frac{1 - \dfrac{1}{x}}{1 + 1\,.\dfrac{1}{x}} \qquad \text{using (5), (6)}$$

hence
$$\frac{7}{9} = \frac{x - 1}{x + 1}$$
$$x = 8$$

Example 2 Simplify $3\sec\left(\tan^{-1}\dfrac{x}{2}\right)$

Let $a = \tan^{-1}\dfrac{x}{2}$, then $\tan a = \dfrac{x}{2}$

$$\cos a = \frac{2}{\sqrt{4 + x^2}}$$

$$\sec a = \frac{1}{\cos a} = \frac{\sqrt{4 + x^2}}{2}$$

$$3 \sec a = \tfrac{3}{2}\sqrt{4 + x^2}$$

TRIGONOMETRICAL EQUATIONS

The simplest but most fundamental type is

(i) **sin x = a**

The **rules** for finding the general solution are:

(a) **Disregard** the **sign** of a and find x in the tables,

(b) Write down the **4** values

x, $180° - x$, $180° + x$, $360° - x$.

(c) Using the sign of a and the **CAST** diagram delete two of these values which lie in the wrong quadrants,

(d) Add any **integer multiple of 360°** to the two remaining values. This gives the **General Solution.**

Example $\cos x = -\frac{1}{2}$.

(a) $x = 60°$,

(b) 60°, 120°, 240°, 300°,

(c) since $\cos x$ is negative, the CAST diagram allows only 120° and 240°.

(d) The general solution is $120° + 360n°$ and $240° + 360n°$ where n is any integer including zero.

(ii) **sin 5x − sin 3x = −sin x.**

The rule for equations of this type is to use the **Factor Formulae.** Hence by (23) we have,

$2 \cos 4x \sin x = -\sin x$

i.e. $\sin x(2 \cos 4x + 1) = 0$

$\sin x = 0$ or $\cos 4x = -\frac{1}{2}$

These two equations are of the first type and the general solution is found as above. Note however that $\cos 4x = -\frac{1}{2}$ has the general solution

$4x = 120° + 360n°, 240° + 360n°$

hence $x = 30° + 90n°, 60° + 90n°$.

In equations of the type $\cos ax = b$, it is vitally important to write down the general values of ax before attempting to find the general values of x itself.

(iii) **cos 2x = 3 sin x + 2**
Convert this type of equation into a quadratic by using (15). This gives

$2\sin^2 x + 3\sin x + 1 = 0$

hence $\sin x = -\frac{1}{2}$ or $\sin x = -1$

General solutions are $x = 360n^\circ + 210^\circ, 360n^\circ + 330^\circ, 360n^\circ + 270^\circ$.

THE STANDARD LINEAR EQUATION
a cos x + b sin x = c.

(iv) Consider the equation $3\sin x + 4\cos x = 1$.
Two methods can be used.

(a) **Half angle Method**

write $t = \tan\dfrac{x}{2}$.

$\therefore\ \tan x = \dfrac{2t}{1-t^2}$ by use of the tan $2A$ formula

$$\left.\begin{aligned} \sin x &= \frac{2t}{1+t^2} \\ \cos x &= \frac{1-t^2}{1+t^2} \end{aligned}\right\} \text{ see page 33.}$$

These substitutions must be remembered.

We have $\dfrac{6t}{1+t^2} + \dfrac{4-4t^2}{1+t^2} = 1$

Hence $5t^2 - 6t - 3 = 0$

from which $t = \tan\dfrac{x}{2}$ can be found. Use (i) to complete the solution.

(b) **Subsidiary Angle Method**
We rewrite the equation in the form $R\sin(x+\alpha) = C$. Using (1),

$R\sin x\cos\alpha + R\cos x\sin\alpha = 3\sin x + 4\cos x = 1$

Comparing coefficients of $\sin x$, $\cos x$ and solving for R and $\tan\alpha$ gives

$R = \sqrt{3^2 + 4^2} = 5$ (positive value taken) and $\tan\alpha = \frac{4}{3}$. Hence $\alpha = 53° 8'$. Substituting in above gives

$5 \sin(x + 53° 8') = 1$

$$x + 53° 8' = \sin^{-1} 0{\cdot}2$$

Use (i) to find the general solution.

Geometric Equivalent

If a rectangular lamina $ABCD$ in which $AB = b$, $BC = a$, is pivoted about A, the height of C above A when AB makes an angle x with the horizontal through A, is given by $a \cos x + b \sin x$. Hence **solving the equation a cos x + b sin x = c** is equivalent to finding when C is at a height of c units above A.

Clearly this is **only possible** if $-\sqrt{a^2 + b^2} \leq c \leq \sqrt{a^2 + b^2}$

With subsidiary angle method, $\sin(x + \alpha) = \dfrac{c}{\sqrt{a^2 + b^2}}$, which must lie between -1 and $+1$.

Example Express $12 \sin x - 5 \cos x$ in the subsidiary angle form and hence find the maximum positive value of the expression. What is the smallest positive value of x which gives this maximum?

$R \sin(x - \alpha) = R \sin x \cos\alpha - R \cos x \sin\alpha = 12 \sin x - 5 \cos x.$

Equating coefficients gives, $R \cos\alpha = 12$ and $R \sin\alpha = 5$.

Squaring and adding $R^2(\sin^2\alpha + \cos^2\alpha) = 169$, $\therefore R = 13$. Dividing

$$\frac{R \sin\alpha}{R \cos\alpha} = \frac{5}{12}, \qquad \tan\alpha = \frac{5}{12}, \qquad \alpha = 22° 37'$$

Thus $12 \sin x - 5 \cos x = 13 \sin(x - 22° 37') = E$ (say)
E is a maximum when $\sin(x - 22° 37') = 1$. Maximum $E = 13$.
This occurs whenever $x - 22° 37' = 90°, 450°, \ldots, (90° + 360n°)$.
$\therefore$ Least value of x to satisfy this is $112° 37'$.
Note that $R \sin(x - \alpha)$ was used in preference to $R \sin(x + \alpha)$ as a minus sign occurred in the example.
If the expression had been $5 \cos x - 12 \sin x$ then $R \cos(x + \alpha)$ would have been appropriate.

SOLUTION OF TRIANGLES

All triangles can be solved using

(i) The **Sine Formula** $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

or (ii) the **Cosine Formula** $a^2 = b^2 + c^2 - 2bc \cos A$.

Note (a) If the data determines one of the ratios in (i) then the Sine Formula will solve the triangle.

(b) The Cosine Formula is not symmetrical and there are separate formulae for b^2 and c^2.

(c) If A is an **obtuse** angle, $\cos A$ is **negative**

(d) Do **not** 'telescope' the terms of the cosine formula.

$5^2 + 3^2 - 2.5.3 \cos 40°$ is **NOT** $(25 + 9 - 30) \cos 40°$.

Logarithmic calculation can be applied directly to the Sine Formula but not to the Cosine Formula. It is often more convenient, therefore, to avoid the Cosine Formula. This can be done as follows.

(i) Given b, c and the included angle A. Use

$$\tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2}$$

Having found $\frac{1}{2}(B - C)$, remember that $B + C = 180° - A$ and hence B and C can be found.

Note that $\frac{1}{2}(B + C) + \frac{1}{2}(B - C) = B$, and $\frac{1}{2}(B + C) - \frac{1}{2}(B - C) = C$.

(ii) Given a, b, and c use one of the formulae

$$\mathbf{\sin \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}}}$$

$$\mathbf{\cos \frac{A}{2} = \sqrt{\frac{s(s - a)}{bc}}}$$

$$\mathbf{\tan \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}}$$

where $2s = a + b + c$.

There are other corresponding formulae

$$\sin \frac{B}{2} = \sqrt{\frac{(s - c)(s - a)}{ca}} \quad \text{etc.}$$

THE HYPERBOLIC FUNCTIONS

We define the hyperbolic sine (sinh) and cosine (cosh) of x by the relations

$$\mathbf{sinh\,x = \tfrac{1}{2}(e^x - e^{-x}) \qquad cosh\,x = \tfrac{1}{2}(e^x + e^{-x})}$$

From which

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

The reciprocal functions cosech x, sech x, coth x are defined in similar fashion.

$\cosh x + \sinh x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) = e^x.$

OSBORN'S RULE

This rule enables us to obtain relationships between the various hyperbolic functions **directly** from corresponding circular ones.
'Replace the circular function in trigonometrical formulae relating general angles by corresponding hyperbolic functions changing the sign of any direct or **implied product** of two sines.'

e.g. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ gives $\cosh 3x = 4\cosh^3 x - 3\cosh x$

but $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ gives $\sinh 3x = 3\sinh x + 4\sinh^3 x$

and $\tan^2\theta$ corresponds to $-\tanh^2 x$.

SERIES FOR sinh x AND cosh x

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}) = \frac{1}{2}\left\{\left(1 + x + \frac{x^2}{2!} + \cdots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}\ldots\right)\right\}$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

similarly $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \cdots$

Note that (i) for real x, $\sinh x$ can take all values, both positive and negative,
(ii) for real x, $\cosh x \geq 1$.

INVERSE HYPERBOLIC FUNCTIONS

If $x = \sinh y$, then $y = \sinh^{-1} x$. Similarly for other functions. If $x = \sinh y$, then $\cosh y = +\sqrt{x^2 + 1}$

hence $e^y = x + \sqrt{x^2 + 1}$

and $y = \sinh^{-1} x = \log\{x + \sqrt{x^2 + 1}\}$

Similarly $\cosh^{-1} x = \log\{x \pm \sqrt{x^2 - 1}\}$

$$\tanh^{-1} x = \tfrac{1}{2} \log \frac{1 + x}{1 - x} \qquad |x| < 1$$

Example 1 Prove from the definitions of $\sinh x$ and $\cosh x$ that $\cosh^2 x = \frac{1}{2}(1 + \cosh 2x)$

$$\begin{aligned}\cosh^2 x &= \tfrac{1}{4}(e^x + e^{-x})^2 = \tfrac{1}{4}(e^{2x} + e^{-2x} + 2)\\ &= \tfrac{1}{2}\{1 + \tfrac{1}{2}(e^{2x} + e^{-2x})\}\\ &= \tfrac{1}{2}(1 + \cosh 2x)\end{aligned}$$

Example 2 Solve for real values of x the equation $\cosh(\log 3x) = \sinh(\log 2x) + \frac{13}{12}$.

$$\begin{aligned}\cosh(\log 3x) &= \tfrac{1}{2}(e^{\log 3x} + e^{-\log 3x})\\ &= \frac{1}{2}\left(3x + \frac{1}{3x}\right)\\ \sinh(\log 2x) &= \tfrac{1}{2}(e^{\log 2x} - e^{-\log 2x})\\ &= \frac{1}{2}\left(2x - \frac{1}{2x}\right)\end{aligned}$$

hence $\dfrac{1}{2}\left(3x + \dfrac{1}{3x}\right) = \dfrac{1}{2}\left(2x - \dfrac{1}{2x}\right) + \dfrac{13}{12}$

$\therefore\ 6x^2 - 13x + 5 = 0$

$$x = \tfrac{1}{2} \quad \text{or} \quad 1\tfrac{2}{3}$$

Calculus

DIFFERENTIAL COEFFICIENTS. DIFFERENTIATION

Given $y = f(x)$, then

$y + \delta y = f(x + \delta x)$,

the right-hand side meaning that x is replaced by $(x + \delta x)$ in $f(x)$.

e.g. if $f(x) = x^3 + 2x$

then $f(x + \delta x) = (x + \delta x)^3 + 2(x + \delta x)$

Hence by subtraction. $\delta y = f(x + \delta x) - f(x)$

The **Differential Coefficient** $\dfrac{dy}{dx}$ is defined as follows

$$\frac{\mathbf{dy}}{\mathbf{dx}} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} \quad \text{or} \quad \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$\dfrac{\mathbf{dy}}{\mathbf{dx}}$ is often called the **derivative** or **derived function.**

Physically $\dfrac{dy}{dx}$ describes the **rate** at which y is **increasing** as x increases and represents the **gradient** of the tangent to the curve $y = f(x)$ at any particular point.

The definition is used to establish, from first principles, the differential coefficients of functions. In practice, only a small number of functions need to be dealt with in this manner, the vast majority being found using the general rules for differentiation.

Example if $y = \sin x$

then $y + \delta y = \sin(x + \delta x)$

hence $\delta y = \sin(x + \delta x) - \sin x$

$= 2\cos(x + \frac{1}{2}\delta x)\sin\frac{1}{2}\delta x$ using (23)

Dividing by δx and rearranging gives

$$\frac{\delta y}{\delta x} = \cos(x + \tfrac{1}{2}\delta x)\,\frac{\sin\frac{1}{2}\delta x}{\frac{1}{2}\delta x}$$

Now as $\delta x \to 0$, $\sin\frac{1}{2}\delta x/\frac{1}{2}\delta x \to 1$ and $\cos(x + \frac{1}{2}\delta x) \to \cos x$, so that

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \cos x.$$

GENERAL RULES FOR DIFFERENTIATION

For convenience, write Du for $\frac{du}{dx}$, etc. Then, if u and v are functions of x, we have

(i) $\mathbf{D(u + v) = Du + Dv}$

(ii) $\mathbf{D(u - v) = Du - Dv}$

(iii) $\mathbf{D(uv) = uDv + vDu}$

In particular, taking $v = k$ (a constant) in (iii) $\mathbf{D(ku) = kDu}$

(iv) $\mathbf{D\left(\frac{u}{v}\right) = \frac{vDu - uDv}{v^2}}$

More complicated combinations are **Functions of Functions** such as log sin x, $(\sin x)^2$. These are differentiated by using

(v) $\mathbf{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$

Note $\frac{dy}{dx}$ is obtained by 'mentally' cancelling du on the right-hand side.

The differentiation is carried out by means of an intermediate substitution e.g. $y = \log \sin x$

write $u = \sin x$ so that $y = \log u$

$\therefore \frac{du}{dx} = \cos x$ and $\frac{dy}{du} = \frac{1}{u}$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos x \times \frac{1}{u} = \frac{\cos x}{\sin x} = \cot x$$

The **function of a function rule** (v) is of fundamental importance, and with practice the substitution can be done mentally.
e.g. $y = \sin^3 5x$. Here we have the cube of the sine of $5x$.
Hence, differentiate the cube, $3 \sin^2 5x$, then the sine, $\cos 5x$, and lastly the $5x$, 5. Rule (v) now tells us to multiply the intermediate results to find dy/dx.

$$\frac{dy}{dx} = 3 \sin^2 5x \times \cos 5x \times 5 = 15 \sin^2 5x \cos 5x$$

The following table gives the most important standard results.

y	$\dfrac{dy}{dx}$	Remarks
constant	0	
x^n	nx^{n-1}	
$\sin x$	$\cos x$	x in radians
$\cos x$	$-\sin x$	
$\tan x$	$\sec^2 x$	
$\cot x$	$-\operatorname{cosec}^2 x$	
$\operatorname{cosec} x$	$-\operatorname{cosec} x \,.\, \cot x$	
$\sec x$	$\sec x \,.\, \tan x$	
e^x	e^x	
$\log x$	$\dfrac{1}{x}$	
$\sin^{-1} \dfrac{x}{a}$	$\dfrac{1}{\sqrt{(a^2 - x^2)}}$	
$\tan^{-1} \dfrac{x}{a}$	$\dfrac{a}{a^2 + x^2}$	
$\sinh x$	$\cosh x$	look carefully at the differences from the corresponding trig. functions
$\cosh x$	$\sinh x$	
$\tanh x$	$\operatorname{sech}^2 x$	
$\sinh^{-1} \dfrac{x}{a}$	$\dfrac{1}{\sqrt{(a^2 + x^2)}}$	
$\tanh^{-1} \dfrac{x}{a}$	$\dfrac{a}{a^2 - x^2}$	

Example 1 $y = 5x^2 + 7 - \dfrac{4}{x} = 5x^2 + 7 - 4 \,.\, x^{-1}$

$\dfrac{dy}{dx} = 10x + 0 + 4 \,.\, x^{-2} = 10x + \dfrac{4}{x^2}$ using (i)

Example 2 $y = x^2 \tan x$

$\dfrac{dy}{dx} = 2x \tan x + x^2 \sec^2 x$ using (iii)

Example 3 $y = \text{cosech } x = \dfrac{1}{\sinh x}$

$$\frac{dy}{dx} = \frac{0 - 1 . \cosh x}{\sinh^2 x} \qquad \text{using (iv)}$$

$$= -\text{cosech } x . \coth x$$

Example 4 $y = \cos^{-1} x$

i.e. $\cos y = x$ also $\sin y = \sqrt{1 - x^2}$

$$-\sin y = \frac{dx}{dy} \qquad \text{(see also implicit functions)}$$

$$\frac{dy}{dx} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{(1 - x^2)}} \cdot$$

HIGHER DERIVATIVES

If for a particular function y, we have found a derivative $\dfrac{dy}{dx}$ this will in turn be a function of x, and can be differentiated again to obtain a **second derivative**, and so on.

Successive derivatives or derived functions are written

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \text{ etc.}$$

The simplest meaning which can be attached to higher derivatives occurs if we consider a particle moving in a straight line so that at time **t** its distance from a fixed point O on the line is **x.**

Then δx represents an increase of distance in time δt, so that $\dfrac{\delta x}{\delta t}$ represents the **average velocity** over a time interval δt, and $\dfrac{\mathbf{dx}}{\mathbf{dt}}$ represents the **velocity at a particular instant.** If this is denoted by v, then $\dfrac{\mathbf{dv}}{\mathbf{dt}}$ or $\dfrac{\mathbf{d^2x}}{\mathbf{dt^2}}$ represents the **rate at which the velocity is changing** or the **acceleration** of the particle.

If we draw a graph to represent how the position of the particle varies with the time, the **gradients** of this graph at various points can be used to draw a **second graph** showing how the **velocity varies with the time.** The value of the gradient of this graph at a particular point will represent the **acceleration of the particle** at the corresponding time.

THE FUNCTIONAL NOTATION

If we write $y = f(x)$, we have seen that $f(2)$ means the value of the function when $x = 2$. **We extend the notation** as follows.

First we can write $\dfrac{dy}{dx} = f'(x), \dfrac{d^2y}{dx^2} = f''(x), \dfrac{d^3y}{dx^3} = f'''(x)$, etc.

An advantage of this is that we can use **f″(3)** to signify:
Differentiate f(x) twice, and substitute 3 for x in the final result.

For example, if $f(x) \equiv x^3 - 7x^2 + 2$

$$f'(x) = 3x^2 - 14x$$

$$f''(x) = 6x - 14$$

and $\qquad f''(3) = 18 - 14 = 4$

Similarly "Solve the equation $f'(x) = 0$", means solving

$3x^2 - 14x = 0,$

and the answers are $x = 0$ or $4\frac{2}{3}$.

SMALL INCREMENTS

Since $\dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$, it follows that if δx is small,

$$\frac{dy}{dx} \doteqdot \frac{\delta y}{\delta x} \qquad \text{or} \qquad \boldsymbol{\delta y \doteqdot \frac{dy}{dx} \,.\, \delta x}$$

Hence, given a small change δx in x, a small change δy in y can be calculated. In many problems, δx appears as a small **error** in the **measurement** of x and then δy is the small **error** arising in the **calculation** of y.
e.g. A circular field has radius 100 m. If the radius is measured in error as 100·02 m, find the error in the calculation of the area of the field.
Let A be the area and r the radius.
$\therefore A = \pi r^2$.

Given that $r = 100$ m, $\delta r = 0{\cdot}02$ m, it is required to find δA.

$$\delta A \doteqdot \frac{dA}{dr} \,.\, \delta r = 2\pi r \,.\, \delta r \tag{1}$$

$$= 2\pi \times 100 \times 0{\cdot}02$$

$$\doteqdot 12{\cdot}6 \text{ m}^2$$

Approximate **percentage** changes can be calculated in the same way. From line (1) we can write

$$\frac{\delta A}{A} \simeq \frac{2\pi r\delta r}{\pi r^2}$$

$$\text{or } \frac{\delta A}{A} \simeq 2.\frac{\delta r}{r}$$

Hence the **fractional increase in area** is **twice** the fractional increase in the radius. Since the **percentage increase** is **fractional increase** × **100,** the **percentage increases** are also in the **ratio 2 : 1.**
In some cases it is convenient to take logarithms before differentiating. If T is the time of oscillation of a simple pendulum,

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

Then $\log T = \log 2\pi + \frac{1}{2}\log l - \frac{1}{2}\log g$

$$\text{and } \frac{1}{T}.\delta t \simeq \qquad -\frac{1}{2}.\frac{1}{g}.\delta g$$

or fractional **increase** in time = **half the fractional decrease** in g.

RATES OF CHANGE

If t is time, $\frac{dy}{dt}$ and $\frac{dx}{dt}$ are the rates of change of y and x with respect to time.

$$\text{Since } \frac{\mathbf{dy}}{\mathbf{dt}} = \frac{\mathbf{dy}}{\mathbf{dx}}.\frac{\mathbf{dx}}{\mathbf{dt}},$$

the rate of change of y can be found if the rate of change of x is known and y is a function of x.
e.g. A sphere has radius 6 m and the radius is increasing at the rate of 4 cm. per sec. Find the rate of change of the volume of the sphere.
Let V be the volume and r the radius.

$\therefore V = \frac{4}{3}\pi r^3.$

Given that $r = 6$m and $\dfrac{dr}{dt} = \dfrac{1}{25}$ m per sec. it is required to find $\dfrac{dV}{dt}$.

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 4\pi r^2 . \frac{dr}{dt}$$

$$= 4\pi . 36 . \tfrac{1}{25} \simeq 18{\cdot}1 \text{ m}^3\text{s}^{-1}.$$

Again, if sand is being tipped from a conveyor belt at 2 m^3 per minute in a conical heap whose height is always equal to its base radius, we can find the **rate at which the height is rising** when the cone is 3 m high.

For $V = \frac{1}{3}\pi r^2 h$
$= \frac{1}{3}\pi h^3$ in this case

$$\therefore \frac{dV}{dh} = \pi h^2 \text{ and } \frac{dV}{dt} = 2.$$

But $\dfrac{dV}{dt} = \dfrac{dV}{dh} . \dfrac{dh}{dt}$, so that

$$\frac{dh}{dt} = \frac{2}{\pi h^2} = \frac{2}{9 \times \pi} \text{ m. min}^{-1} = \frac{2}{9 \times 60 \times \pi} \text{m.s}^{-1} = 0{\cdot}0012 \text{ m.s}^{-1}.$$

Acceleration as a rate of change.

We have $\dfrac{\mathbf{d^2x}}{\mathbf{dt^2}} = \dfrac{dv}{dt} = \dfrac{dv}{dx} . \dfrac{dx}{dt} = \mathbf{v}\dfrac{\mathbf{dv}}{\mathbf{dx}}$.

This form of the acceleration is of great importance.

TANGENTS AND NORMALS

Given a curve $y = f(x)$ and a point $P(x_1, y_1)$ on it. The **tangent** at P has the equation $(y - y_1) = m(x - x_1)$, where m is the gradient of the tangent at P. i.e. $m = dy/dx$ at (x_1, y_1). Remember to substitute $x = x_1$ and $y = y_1$ into dy/dx to find m.

The **normal** at P is $y - y_1 = m'(x - x_1)$ where m' is the gradient of the normal.
Since the normal is perpendicular to the tangent
$\mathbf{mm' = -1}$ (see p. 67).

Hence first find m as above and then m' by using $m' = -\dfrac{1}{m}$

For example, the gradient of $y = x^3 - 2x^2 + 4x - 3$ at the point $(1, 0)$ is given by $\dfrac{dy}{dx} = 3x^2 - 4x + 4$.

When $x = 1$ we have $\dfrac{dy}{dx} = 3$

$\therefore$ The **tangent** is $y - 0 = 3(x - 1)$
i.e. $\mathbf{y = 3x - 3.}$
The **normal** has a gradient $-\frac{1}{3}$, so that its equation is
$y - 0 = -\frac{1}{3}(x - 1)$
i.e. $\mathbf{3y + x = 1}$

IMPLICIT FUNCTIONS

When the equation of a curve is of the form $F(x, y) = 0$ then it is not necessary, even if it is possible, to express y in terms of x, i.e. in the form $y = f(x)$. The method used is as follows,

Take $y + \sin y = x$ (1)

Since $\sin y$ is a function of y and y is a function of x, we apply the rule for differentiating a function of a function.

This gives $\dfrac{d}{dx}(\sin y) = \dfrac{d}{dy}(\sin y) \times \dfrac{dy}{dx} = \cos y \dfrac{dy}{dx}$

Hence (1) differentiated term by term w.r.t.x. gives

$$\frac{dy}{dx} + \cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{1 + \cos y}$$

Example Find the equations of the tangent and normal to the curve $x^3 - 3xy^2 + y^3 + 19 = 0$ at the point (2, 3).
Differentiating w.r.t.x gives

$$3x^2 - 3x \, . \, 2y\frac{dy}{dx} - 3y^2 + 3y^2\frac{dy}{dx} = 0$$

At the point (2, 3), after substitution in the above, $\dfrac{dy}{dx} = -\dfrac{5}{3}$

$\therefore$ the tangent is $y - 3 = -\frac{5}{3}(x - 2)$

$3y = 19 - 5x$

and the normal is $y - 3 = \frac{3}{5}(x - 2)$

$5y = 3x + 9$

PARAMETRIC EQUATIONS

If the equation of a curve is given by a pair of **parametric equations**

$y = f(t)$ and $x = g(t)$

where **t** is a **variable** called a **parameter,** then we can still find the gradient by differentiation.
Let δt be a small increment in t and δx and δy be the corresponding increments in x and y.

We may write $\dfrac{\delta y}{\delta x} = \dfrac{\delta y/\delta t}{\delta x/\delta t}$

Now as $\delta t \to 0$, $\delta x \to 0$ and $\delta y \to 0$ and we obtain the limit,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{f'(t)}{g'(t)} \qquad (g'(t) \neq 0)$$

If the second derivative is needed it may be found from the formula

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \bigg/ \frac{dx}{dt}.$$

Example Find the equation of the tangent to the curve given by $x = 2\cos 2t$, $y = 3\sin 2t$, at the point where $t = \pi/6$.

When $t = \pi/6$, $x = 2\cos \pi/3 = 1$ and $y = 3\sin \pi/3 = (3\sqrt{3})/2$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{6\cos 2t}{4\sin 2t}$$

$$= \frac{-3}{2\sqrt{3}} \qquad \text{when } t = \pi/6$$

$$= -\frac{\sqrt{3}}{2}$$

the tangent is $y - \frac{3\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}(x - 1)$

$2y + \sqrt{3}x - 4\sqrt{3} = 0$

MAXIMA, MINIMA AND INFLEXIONS

A **maximum turning point** on a curve is where a curve reaches its highest point and begins to come down again.

A **minimum turning point** is where a curve reaches its lowest point and begins to rise again.

In both cases, the tangent is horizontal,

i.e. $\frac{dy}{dx} = 0$.

Solving this equation gives the possible positions of the maxima and minima. Suppose that x_1 is one such point.

Consider the **sign of f**$'(x)$ for values of x slightly less than x_1 and for values of x slightly greater than x_1.

If $f'(x)$ changes **from** + through 0 **to** −, the value x_1 gives a **maximum** value of the function.

If $f'(x)$ changes **from** − through 0 **to** +; the value x_1 gives a **minimum** value of the function.

Points at which the **gradient** of the graph of a function **has a maximum or minimum value** are called **points of inflexion.** These are points where the **tangent crosses the curve,** and are found by solving the equation $f''(x) = 0$ unless $f'(x) = 0$ also.

An alternative method of distinguishing between maxima and minima

is to substitute $x = x_1$ in $\dfrac{d^2y}{dx^2}$.

If (i) $\dfrac{d^2y}{dx^2}$ at $x = x_1$ is **negative,** the point is a **maximum.**

(ii) $\dfrac{d^2y}{dx^2}$ at $x = x_1$ is **positive,** the point is a **minimum.**

If $\dfrac{d^2y}{dx^2} = 0$ at $x = x_1$, then find the value of $\dfrac{d^3y}{dx^3}$ at $x = x_1$. If this is **not zero** then the point is an **inflexion.**

Example Find the turning points of the curve $y = x^3 - 7x^2 + 8x + 3$ and distinguish between them. Find also where an inflexion occurs.

Let $f(x) = x^3 - 7x^2 + 8x + 3$, then

$f'(x) = 3x^2 - 14x + 8$

$\quad = (3x - 2)(x - 4)$

Hence $f'(x) = 0$ when $x = \frac{2}{3}$ or 4.

We may proceed to identify the turning points in two ways,

(i) If $x < \frac{2}{3}$, $f'(x)$ is $(-)(-)$, i.e. >0

If $\frac{2}{3} < x < 4$, $f'(x)$ is $(+)(-)$, i.e. <0

If $x > 4$, $f'(x)$ is $(+)(+)$, i.e. >0

Hénce $x = \frac{2}{3}$ gives a **maximum,** and $x = 4$ gives a **minimum.**

(ii) $f''(x) = 6x - 14$

$f''(\frac{2}{3}) = 6 \times \frac{2}{3} - 14 < 0$. A maximum point.

$f''(4) = 6 \times 4 - 14 > 0$. A minimum point.

The maximum value is $f(\frac{2}{3}) = 4\frac{23}{27}$ and the minimum is $f(4) = -13$.
The point of inflexion occurs where $f''(x) = 0$ (and $f'''(x) \neq 0$).

i.e. where $6x - 14 = 0$, $x = 2\frac{1}{3}$. $(f'''(2\frac{1}{3}) \neq 0)$

Note. A curve may have several maxima and several minima as we mean by these terms that they are **locally** highest and lowest points on the curve.

There must be a minimum between each pair of maxima and *vice versa* and there must be an inflexion between each maximum and minimum unless the curve is discontinuous.

LEIBNITZ'S THEOREM

This theorem tells us how to differentiate a product **n times** provided each term of the product can be so differentiated. Let u_n and v_n be the nth derivatives of u and v. Then Leibnitz says, $\mathbf{(uv)_n} =$

$$\mathbf{u_n v + C_1 u_{n-1} v_1 + C_2 u_{n-2} v_2 + \cdots + C_r u_{n-r} v_r + \cdots + u v_n}$$

Note that the coefficients are the same as those in the binomial theorem and that the products follow a similar pattern to the binomial ones.

Example Find the nth derivative of x^3e^x.
Take $v = x^3$, $u = e^x$. Then $v_1 = 3x^2$, $v_2 = 6x$, $v_3 = 6$, $v_4 = v_5 = \cdots = 0$.
$u_1 = u_2 = \cdots u_n = e^x$.

$$\frac{d^n}{dx^n}(x^3e^x) = e^x(x^3 + 3nx^2 + 3n(n-1)x + n(n-1)(n-2))$$

TAYLOR'S THEOREM

This theorem shows how to expand a function $f(x + h)$ as an infinite series.

Let $f(x + h) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ (A)

Repeated differentiation gives

$$f'(x + h) = a_1 + 2a_2x + 3a_3x^2 + \cdots$$

$$f''(x + h) = 2a_2 + 3.2.a_3x + \cdots$$

$$f'''(x + h) = 3.2.a_3 + \cdots$$

If we now set $x = 0$ in all the above, we obtain $f(h) = a_0$, $f'(h) = a_1$, $f''(h) = 2a_2$, $f'''(h) = 3.2.a_3$. In general the rth derivative $f^r(h) = r!\,a_r$. Substituting in (A) gives **Taylor's theorem,**

$$\mathbf{f(x + h) = f(h) + xf'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \cdots}$$

Example Use Taylor's theorem to find tan 45°12′ to 5 decimal places.

Write $f(h) = t = \tan h$ (h in radians)

Then $f'(h) = \sec^2 h = 1 + \tan^2 h = 1 + t^2$

$$f''(h) = 2t(1 + t^2) = 2t + 2t^3$$

$$f'''(h) = 2(1 + 3t^2)(1 + t^2) = 2(1 + 4t^2 + 3t^4)$$

$$f\left(\frac{\pi}{4}\right) = 1, \quad f'\left(\frac{\pi}{4}\right) = 2, \quad f''\left(\frac{\pi}{4}\right) = 4, \text{ etc.}$$

also $x = 12' = \dfrac{12}{60} \times \dfrac{\pi}{180}$ radian $\simeq 0{\cdot}00349066$

Substituting in the series gives

$$\begin{aligned}\tan 45^\circ\, 12' &= 1 + 0{\cdot}00698132 + 0{\cdot}00002437 + 0{\cdot}00000006 + \ldots \\ &= 1{\cdot}00700575 \doteqdot 1{\cdot}00701\end{aligned}$$

MACLAURIN'S THEOREM

This theorem is a special case of Taylor's theorem. Setting $h = 0$ in it gives Maclaurin's theorem.

$$\mathbf{f(x) = f(0) + xf'(0) + \frac{x^2}{2!}\, f''(0) + \frac{h^3}{3!}\, f'''(0) + \cdots}$$

Some important Maclaurin expansions are

(i) $\mathbf{e^x} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots$

(ii) $\mathbf{\sin x} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots$

(iii) $\mathbf{\cos x} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots$

(iv) $\mathbf{\log(1 + x)} = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$

(v) $\mathbf{\log(1 - x)} = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \cdots$

(vi) $\mathbf{(1 + x)^n} = 1 + nx + \dfrac{n(n-1)}{2!}\, x^2 + \dfrac{n(n-1)(n-2)}{3!}\, x^3 + \ldots$

(Binomial Theorem)

Note that (ii) and (iii) show that, for small x

$\sin x \simeq x$ and $\cos x \simeq 1 - \frac{1}{2}x^2$.

Remember that (iv) is only true for $-1 < x \leq 1$

(v) is only true for $-1 \leq x < 1$

If n is not a positive integer, (vi) is only true for $-1 < x < 1$.

These expansions are often used to find approximate values of a function for small values of x.
Composite functions are best dealt with by using the formulae (i) to (vi) rather than by reference to the theorem itself.

Thus $\log(1+\sin x) = \sin x - \frac{1}{2}(\sin^2 x) + \frac{1}{3}(\sin^3 x)\ldots$

$$= \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right) - \frac{1}{2}\left(x - \frac{x^3}{6}\right)^2 + \frac{1}{3}\left(x - \frac{x^3}{6}\right)^3$$

$$= x - \frac{x^3}{6} + \cdots - \frac{x^2}{2} + \frac{x^4}{6} \cdots + \frac{x^3}{3},$$

neglecting terms above x^4. Hence

$$\log(1+\sin x) \simeq x - \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{6}$$

Example Use Maclaurin's series to verify that the expansion of $\sec x$ is $1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots$

$f(x) = \sec x$ $\qquad f(0) = 1$

$f'(x) = \sec x \tan x$ $\qquad f'(0) = 0$

$f''(x) = \sec x \tan^2 x + \sec^3 x$ $\qquad f''(0) = 1$

$f'''(x) = \sec x \tan^3 x + 2\tan x \sec^3 x + 3\sec^3 x \tan x$ $\qquad f'''(0) = 0$

$f''''(x) = \sec x \tan^4 x + 3\tan^2 x \sec^3 x + 2\sec^5 x + 6\sec^3 x \tan^2 x + 9\sec^3 x \tan x + 3\sec^5 x.$ $\qquad f''''(0) = 5$

Substituting gives

$$\sec x = 1 + 0 + \tfrac{1}{2}x^2 + 0 + \frac{5}{4!}x^4 + \cdots$$

$$= 1 + \tfrac{1}{2}x^2 + \tfrac{5}{24}x^4 + \cdots$$

Note that certain functions cannot be expanded by Maclaurin's theorem. For instance, $\log x$ is not defined when $x = 0$ and any derivative of the form $1/x^n$, when $n > 0$, requires division by zero which again is not defined.

INTEGRALS

If $f'(x) = z$, $f(x)$ is a function which when differentiated gives $z.f(x)$ is called **an integral of z with respect to x,** and we write $f(x) = \int z \, dx$.
The process of finding $\int f(x) \, dx$ involves **finding a function which, when differentiated, gives f(x).**
Basically, the only method of solving this problem is by **recognition.** It is vital that the table of differential coefficients be memorised. They lead to the following table:

y	$\int y \, dx$
$x^n \ (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$
$\dfrac{1}{x}$	$\log x$ (provided $x > 0$)
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\operatorname{cosec}^2 x$	$-\cot x$
e^x	e^x
$\dfrac{1}{\sqrt{(a^2 - x^2)}}$	$\sin^{-1} \dfrac{x}{a}$
$\dfrac{1}{a^2 + x^2}$	$\dfrac{1}{a} \tan^{-1} \dfrac{x}{a}$
$\dfrac{1}{\sqrt{(a^2 + x^2)}}$	$\sinh^{-1} \dfrac{x}{a}$

It must be appreciated that the table shows only a few of the standard results. Note that each of the above integrals should have a **constant of integration** added. This is due to the fact that constants disappear when we differentiate and when the process is reversed there is no way of knowing the magnitude of the missing constant unless further information is supplied. Integrals in which the constant is unknown are called **indefinite integrals.**
Expressions for indefinite integrals obtained by differing methods often seem different; this is due to the results having different constants, the variable parts of the integrals will be identical.

The 'methods' of integration which we now consider are means of getting the integrand (i.e. $f(x)$) into a recognisable form.

(i) **Direct change of the integrand**

$$\int \frac{x^2+1}{x}\,dx = \int \left(x + \frac{1}{x}\right) dx$$

$$\int \frac{x+1}{\sqrt{x}}\,dx = \int (x^{1/2} + x^{-1/2})\,dx$$

$$\int \frac{x+3}{x^2-1}\,dx = \int \left(\frac{2}{x-1} - \frac{1}{x+1}\right) dx$$ see page 21

$$\int \sin^2 x\,dx = \tfrac{1}{2}\int (1 - \cos 2x)\,dx$$

$$\int \sin 2x \cos x\,dx = \tfrac{1}{2}\int (\sin 3x + \sin x)\,dx$$

$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx$$ see (16) page 32

(ii) **Change of the integrand by substitution**

$\mathrm{I} = \int \tan x \sec^2 x\,dx.$

Write $u = \tan x$, hence $\mathbf{du = sec^2\,x\,.\,dx.}$

$\therefore\ \mathrm{I} = \int u\,du = \tfrac{1}{2}u^2 = \tfrac{1}{2}(\tan x)^2 + c.$

$$\mathrm{I} = \int \frac{x^2\,dx}{\sqrt{1-x^2}}$$

write $x = \sin\theta$, hence $\mathbf{dx = cos\,\theta\,d\theta.}$

$$\therefore\ \mathrm{I} = \int \frac{\sin^2\theta}{\cos\theta}.\cos\theta\,d\theta = \int \sin^2\theta\,d\theta \text{ and so on.}$$

Remember (a) to **substitute for dx**
(b) to **change back to x after integration.**

(iii) **Change of the integrand by parts**

The integral of a product $\int u\,.\,v\,dx$ can be changed by the formula

$$\int u\,.\,v\,dx = v\int u\,dx - \int \frac{dv}{dx}\left(\int u\,dx\right) dx.$$

The right-hand side is best remembered in **words.** 'The integral of the first × the second **minus** the integral of the integral of the first × the differential coefficient of the second', u being referred to as the first and v as the second.
This method is very useful when the integral contains an inverse function such as log x, *tan*$^{-1}$ x, etc.

e.g. $\int x \tan^{-1} x \, dx = \frac{x^2}{2} . \tan^{-1} x - \int \frac{x^2}{2} . \frac{1}{1 + x^2} \, dx.$

It is useful sometimes to take $u = 1$.

e.g. $\int \log x \, dx = \int 1 . \log x \, dx = x \log x - \int x . \frac{1}{x} \, dx$

(iv) **A useful form of the integrand to remember is when the numerator is the differential coefficient of the denominator**

The integral is then **log (denominator).**

e.g. $\int \frac{2x}{x^2 + 1} \, dx = \log (x^2 + 1)$

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \log \sin x.$$

All the integrals above are **indefinite integrals** and the answer should include the **addition of an arbitrary constant k.**

e.g. $\int \frac{2x}{x^2 + 1} \, dx = \log (x^2 + 1) + k.$

DEFINITE INTEGRALS. AREAS AND VOLUMES

The integral becomes a **definite integral** if **limits** are placed on the integral sign, $\int_a^b$

The integral then has a definite **numerical** value found by substituting **a** and **b** in the answer and subtracting the **former from the latter.** No k is necessary.

e.g. $\int_a^b \frac{2x}{x^2 + 1} \, dx = \Big[\log (x^2 + 1)\Big]_a^b$

$$= \log (b^2 + 1) - \log (a^2 + 1)$$

If a curve $y = f(x)$ is drawn between the values $x = a$ and $x = b$, the **area** between the ordinates $x = a$, $x = b$, the curve and the $x =$ axis is

$$\int_a^b f(x)\,dx$$

If the curve $y = f(x)$ is rotated about the x-axis so as to form a complete volume of revolution, the **volume** so formed is

$$\int_a^b \pi y^2\,dx \quad \text{or} \quad \int_a^b \pi\{f(x)\}^2\,dx$$

For example, if the area between the parabola $y = 9 - x^2$ and the x-axis is rotated about that axis, we have:
(1) $y = 9 - x^2$ cuts $y = 0$, where $x = \pm 3$, and therefore:

$$(2)\ V = \int_{-3}^{+3} \pi(9 - x^2)^2\,dx$$

$$= \pi \int_{-3}^{+3} (81 - 18x^2 + x^4)\,dx$$

$$= \pi \left[81x - 6x^3 + \frac{1}{5}x^5\right]_{-3}^{+3} = \frac{1296\pi}{5}$$

Again, to find the area between $y = x^2 - 6x + 15$ and $y = 9 + x$, we first **find the points of intersection.**

They meet where $x^2 - 6x + 15 = 9 + x$

when $x^2 - 7x + 6 = 0$, i.e. when $x = 1$ or 6.

$$\text{Area} = \int_1^6 (9 + x)\,dx - \int_1^6 (x^2 - 6x + 15)\,dx$$

$$= \int_1^6 (-x^2 + 7x - 6)\,dx$$

$$= \left[-\frac{x^3}{3} + \frac{7x^2}{2} - 6x\right]_1^6 = 20\tfrac{5}{6}.$$

There is a common trap for the unwary in dealing with areas. Consider finding the area enclosed between the curve $y = \frac{1}{3}x^2 - 3$ the ordinates at $x = 9$ and $x = 1$.

Proceeding as before, Area $= \int_1^9 (\frac{1}{3}x^2 - 3)\,dx$

$$= \left[\frac{x^3}{9} - 3x\right]_1^9$$

$$= (81 - 27) - (\tfrac{1}{9} - 3) = 56\tfrac{8}{9}$$

Now make a sketch of the curve and see that it cuts the x-axis at $x = 3$. Let us now repeat the calculation but taking the area in two parts, for $x < 3$ and $x > 3$.

$$\text{Area} = \int_3^9 \left(\frac{x^2}{3} - 3\right) dx + \int_1^3 \left(\frac{x^2}{3} - 3\right) dx$$

$$= 60 + (-3\tfrac{1}{9})$$

The trap is now apparent. The area below the x-axis has a **negative value.** The area required is therefore $60 + 3\frac{1}{9} = 63\frac{1}{9}$.
To find the area between $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$, when the curve cuts the x-axis at $x = c$, and c lies between a and b, divide the integration thus:

$$\int_b^a f(x)\,dx = \int_c^a f(x)\,dx + \int_b^c f(x)\,dx \quad (b < c < a)$$

taking each part as positive.

DIFFERENTIAL EQUATIONS

A Differential Equation is an equation containing differential coefficients and its solution involves finding an expression for y in terms of x.

Variables separable

The simplest type of differential equation is known as variables separable because all the terms involving y can be moved to one side of the equation and all the terms involving x (including dx) can be moved to the other.

e.g. $\dfrac{dy}{dx} = xy$

$$\therefore \frac{dy}{y} = x\,dx$$

$$\therefore \int \frac{dy}{y} = \int x\,dx$$

$$\therefore \log y = \tfrac{1}{2}x^2 + k$$

Note the constant k which **has to be written in.** In a practical problem a **Boundary Condition** is given. That is, corresponding values of x and y are given from which k can be found.
e.g. Given $y = 1$ when $x = 4$

$$k = \log 1 - \frac{16}{2} = -8$$

Integrating Factor

Equations of the type $\frac{dy}{dx} + Py = Q$ where P,Q are functions of x may be solved by multiplying each side of the equation by a suitably chosen function.
This Integrating Factor can sometimes be found by inspection but **E$\int$ Pdx** is always a suitable factor. It is evaluated **without** the addition of a constant of integration.

Example Solve the equation $\frac{dy}{dx} + \cot x \,.\, y = x$

The I.F. is $e^{\int \cot x\,dx} = e^{\log \sin x} = \sin x$

The equation becomes

$$\sin x \frac{dy}{dx} + \sin x \,.\cot x \,.\, y = x \,.\sin x$$

$$\frac{d}{dx}(y \,.\sin x) = x \,.\sin x$$

Integrating the R.H.S. by parts

$$\int x \,.\sin x\,dx = -x \,.\cos x - \int -\cos x\,dx = -x \,.\cos x - \sin x + k$$

$$\therefore y \sin x = -x \,.\cos x - \sin x + k$$

$$(y + 1)\sin x = -x \,.\cos x + k$$

SECOND ORDER EQUATIONS

Second order equations are those with second derivatives and none higher. We shall only consider those with constant coefficients.

$$\mathbf{\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0} \qquad a \text{ and } b \text{ are constants}$$

We try the solution $\mathbf{y = e^{mx}}$; then $\dfrac{dy}{dx} = me^{mx}, \dfrac{d^2y}{dx^2} = m^2e^{mx}$

Thus $m^2 + am + b = 0$

This equation has two roots m_1 and m_2. Three possibilities must be considered.

(i) Real distinct roots. The general solution is

$$\mathbf{y = Ae^{m_1x} + Be^{m_2x}}$$

(ii) Real equal roots. The general solution is

$$\mathbf{y = (A + Bx)e^{mx}}$$

(iii) Complex roots. $m_1 = (p + iq)$, $m_2 = (p - iq)$. The general solution is

$$\mathbf{y = e^{px}(A \cos qx + B \sin qx)}$$ (see p. 87 *et seq.*)

A and B are constants whose values can be determined from the boundary or initial conditions.

Example The S.H.M. equation $\dfrac{d^2x}{dt^2} + \omega^2x = 0$ is an important case.

Putting $x = e^{mt}$ leads to

$$m^2 + \omega^2 = 0$$

$$m = \pm\omega i$$

$$x = A \cos \omega t + B \sin \omega t$$

Coordinate Geometry

COORDINATES AND LOCI

Ox and Oy are two coplanar straight lines perpendicular to each other. P is a point in the plane of these lines and PX is the perpendicular from P to Ox. Clearly OX and PX are sufficient to **locate** P uniquely.
OX is called the **x-coordinate** or **Abscissa** of P and PX is called the **y-coordinate** or **Ordinate** of P.
Writing $OX = x$ and $PX = y$, we say that P has **coordinates** (x, y). Such coordinates are called **Rectangular Cartesian coordinates.**
The **distance** d between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by using Pythagoras's Theorem. Clearly

$$\mathbf{d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2}$$

The coordinates of the point R dividing the line PQ **internally** in the ratio $l : m$ are

$$\left(\frac{mx_1 + lx_2}{l + m}, \quad \frac{my_1 + ly_2}{l + m}\right)$$

In particular, if $l = m$, R is the **mid-point** of PQ and its coordinates are

$$\left(\frac{x_1 + x_2}{2}, \quad \frac{y_1 + y_2}{2}\right) \tag{1}$$

The coordinates of the point R dividing the line PQ **externally** in the ratio $l : m$ are

$$\left(\frac{mx_1 - lx_2}{m - l}, \quad \frac{my_1 - ly_2}{m - l}\right)$$

If the vertices A, B, C of a triangle are (x_1, y_1), (x_2, y_2), (x_3, y_3) the mid-point F of AB is given by equations (1).
The point dividing FC in the ratio $1 : 2$ is therefore

$$\left[\left(2.\frac{x_1 + x_2}{2} + x_3\right) \div 3, \quad \left(2.\frac{y_1 + y_2}{2} + y_3\right) \div 3\right]$$

that is $\left(\dfrac{x_1 + x_2 + x_3}{3}, \quad \dfrac{y_1 + y_2 + y_3}{3}\right)$

The **symmetry** of this result shows that if points were chosen in the same way on the other **medians** the same final position would be

reached. Hence the **medians of a triangle meet at a point G such that CG : GF = 2 : 1.**

Given the coordinates of the vertices of a triangle, to find the area of the triangle.

Draw the ordinates through each vertex of the triangle. The area of the triangle may then be found from the areas of the three trapezia so formed. Alternatively, let the triangle have coordinates (x_1, y_1), (x_2, y_2) (x_3, y_3) and apply the above procedure. Then after simplifying we have

$$\textbf{Area} = \tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3)$$

If the coordinates are taken in clockwise order the answer is negative, if taken anticlockwise the answer is positive.

LOCI

If the point P with coordinates (x, y) moves in the plane of the axes so that its position is **governed by a given condition,** the **path** that P traces out is called the **Locus of P.**
If an equation is written down which includes x and y and **no other variable** and which satisfies the given condition, then this equation is the equation of the Locus.
e.g. If P moves so that its distance from the x-axis is always twice its distance from the y-axis, then this condition stated in terms of x and y is

$$y = 2x$$

and this **is** the **equation of the locus of P.**
Points whose distance from the x-axis is zero are clearly given by

$$\mathbf{y = 0}$$

which is therefore the **equation of the x-axis.**
Similarly, the y-axis is the line

$$x = 0$$

Sometimes the given condition is expressed in terms of a third variable which is called a **Parameter.**
e.g. If P moves so that $x = t^2$ and $y = 2t$, then x and y are expressed in terms of the parameter t.
The equation in x and y which **does not** include t is found by eliminating t.

$$x = t^2 = \left(\frac{y}{2}\right)^2$$

i.e. $4x = y^2$

and this, therefore, is the equation of the locus of P.
If A, C are the points (a, b), (c, d), the locus of points P which are equidistant from A and C is given by

$$\sqrt{(x-a)^2 + (y-b)^2} = \sqrt{(x-c)^2 + (y-d)^2}$$

Squaring both sides gives

$$(x-a)^2 + (y-b)^2 = (x-c)^2 + (y-d)^2$$

which reduces to

$$2(c-a)x + 2(d-b)y = c^2 + d^2 - a^2 - b^2.$$

This is the equation of the **perpendicular bisector** of AC.
Again the locus of points distant **a** units from a **fixed point** (h, k) is **the circle**

$$(x-h)^2 + (y-k)^2 = a^2.$$

THE STRAIGHT LINE

Suppose that a straight line cuts the y-axis at $(0, c)$ and is inclined at α to the x-axis. Let $P(x, y)$ be on the line, then

$$\tan \alpha = \frac{y-c}{x}$$

i.e. $\mathbf{y = mx + c}$, where $m = \tan \alpha$.
This is an equation **connecting x and y and no other variable.**
It is, therefore, the **equation of the line.**
By multiplying through by a constant, it follows that the equation of a line can be written in the form

$$ax + by + c = 0$$

Remember, **y = mx + c is the most convenient form.**

The equation of a line through a given point with given slope.

Let the given point be $Q(x_1, y_1)$ and let $P(x, y)$ be any other point on the line.

Then slope of $PQ = m = \dfrac{y - y_1}{x - x_1}$

$\therefore \mathbf{y - y_1 = m(x - x_1)}$

To find the equation of a line through two given points A and B

Let A be (2, 3), B be (4, 5) and the line be $y = mx + c$.
These pairs of coordinates must satisfy the equation of the line. Hence

$3 = 2m + c$

$5 = 4m + c$

from which m and c can be found.
More generally, if A is (x_1, y_1) and B is (x_2, y_2), the above method shows that the equation of the line is

$$\frac{\mathbf{y - y_1}}{\mathbf{y_2 - y_1}} = \frac{\mathbf{x - x_1}}{\mathbf{x_2 - x_1}}$$

This result can also be obtained by using the device used in the previous section. The slope of the line joining A and B is $\dfrac{y_2 - y_1}{x_2 - x_1} = m$. Substitute in the above to give

$$\mathbf{y - y_1} = \frac{\mathbf{y_2 - y_1}}{\mathbf{x_2 - x_1}}(\mathbf{x - x_1})$$

from which the result follows.

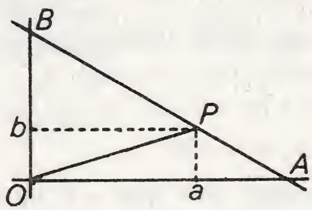
The Intercept Form

If a line cuts off intercepts of lengths a and b on the x and y axes, and $P(x, y)$ is any point on the line, then

$\triangle OPB + \triangle OAP = \triangle OAB$

i.e. $\frac{1}{2}bx + \frac{1}{2}ay = \frac{1}{2}ab$

so that the locus is $\dfrac{\mathbf{x}}{\mathbf{a}} + \dfrac{\mathbf{y}}{\mathbf{b}} = \mathbf{1}$



The Perpendicular Form

If the length of the perpendicular from O, $(0, 0)$ to the line is **p** units, and the perpendicular makes an angle α with OX, then the **intercepts** of the line with the axes are **p sec α** and **p cosec α.**

so that the equation is $$\frac{x}{p \sec \alpha} + \frac{y}{p \operatorname{cosec} \alpha} = 1$$

or $$\mathbf{x \cos \alpha + y \sin \alpha = p.}$$

Angle between lines

Suppose two lines are inclined at angles α_1, α_2 to the x-axis, where $\tan \alpha_1 = m_1$ and $\tan \alpha_2 = m_2$.
The angle between the lines is $(\alpha_1 - \alpha_2)$ and

$$\tan(\alpha_1 - \alpha_2) = \frac{\tan \alpha_1 - \tan \alpha_2}{1 + \tan \alpha_1 \tan \alpha_2}$$ see (6) on page 32

∴ the **angle between the lines** is

$$\mathbf{\tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right).}$$

Parallel lines

The condition for these is clearly $\mathbf{m_1 = m_2}$.

Perpendicular lines

If a line is inclined to the x-axis at α, then $m = \tan \alpha$. A line perpendicular to the given line cuts the x-axis at $(90^\circ + \alpha)$. Hence its gradient m' is given by

$$m' = \tan(90^\circ + \alpha) = -\cot \alpha = -\frac{1}{m}$$

Hence the **condition for perpendicularity** is

$\mathbf{m_1 m_2 = -1.}$

There is a very simple way of **writing down** the equation of a **line perpendicular to another.**

Consider the equations $\mathbf{ax + by = c}$

$$\mathbf{bx - ay = d}$$

The gradient of the first is $-\dfrac{\mathbf{a}}{\mathbf{b}}$ and that of the second $\dfrac{\mathbf{b}}{\mathbf{a}}$.

But $-\dfrac{a}{b} \times \dfrac{b}{a} = -1$, so that **the lines are perpendicular.**

Example Write down the equation of the line through $(5, -1)$ which is perpendicular to $2x + 3y = 0$.

Any perpendicular to $\mathbf{2x + 3y = 0}$ has an equation of the form

$$\mathbf{3x - 2y = C.}$$

The point $(5, -1)$ will lie on this if

$3.5 - 2(-1) = C$, or $C = 17$.

With practice we can write down the equation in one step. The perpendicular through $(-4, 1)$ will be

$$\begin{aligned} 3x - 2y &= 3(-4) - 2(1) \\ &= -14 \end{aligned}$$

Similarly the line through $(5, -1)$ parallel to

$$2x + 3y = 0$$

is $2x + 3y = 2.5 + 3(-1)$

or $2x + 3y = 7$.

Perpendicular Distance and Angle Bisectors

Given the line $\mathbf{ax + by + c = 0}$

and the point $P\ (h, k)$. The **perpendicular distance p** from P to the line is given by

$$\mathbf{p} = \pm \frac{\mathbf{ah + bk + c}}{\sqrt{\mathbf{a}^2 + \mathbf{b}^2}}$$

When finding a numerical value for p, its **positive value** is taken. **Note** carefully that the equation of the line must be written **as shown** before substituting the coordinates of P to obtain the numerator.
The **perpendicular form** $x \cos\alpha + y \sin\alpha - p = 0$ gives an even simpler form for the **length of the perpendicular.**
If p' is the length of the perpendicular from (h, k) to the line

$$x \cos\alpha + y \sin\alpha - p = 0 \qquad (1)$$

then $\mathbf{p'} = \pm(\mathbf{h} \cos \alpha + \mathbf{k} \sin \alpha - \mathbf{p})$.

Note the transposition of equation (1) from the form given on page 67.

The linear equation **must be written with all terms on one side.**

Given two lines $a_1x + b_1y + c_1 = 0$ (2)

$$a_2x + b_2y + c_2 = 0 \quad (3)$$

If $P(x, y)$ satisfies the condition that the perpendicular distances from P to the lines are equal, then

$$\frac{\mathbf{a_1x + b_1y + c_1}}{\sqrt{\mathbf{a_1}^2 + \mathbf{b_1}^2}} = \pm \frac{\mathbf{a_2x + b_2y + c_2}}{\sqrt{\mathbf{a_2}^2 + \mathbf{b_2}^2}}$$

These, then, are the **equations** of the internal and external **bisectors** of the angles between the two lines. To identify which is which, compare the gradients of these bisectors with the gradients of the given lines.

$\mathbf{L_1 + kL_2 = 0}$.

If $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are the equations of two straight lines, then the equation

$$\mathbf{(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0} \quad (4)$$

represents for different numerical values of k, **a set of straight lines** all passing through the **intersection** of the lines (2) and (3). For (a) the equation is of the **first degree,** and (b) the point whose co-ordinates satisfy equations (2) and (3) **must also satisfy (4).** Notice the simple way of dealing with the problem which follows.

Example Find the line which passes through the origin and also passes through the intersection of $3x + 4y - 2 = 0$ and $5x - 7y + 3 = 0$. Any line through the point of intersection of the given lines is

$(3x + 4y - 2) + k(5x - 7y + 3) = 0$

If this line passes through the origin (0, 0)

$3(0) + 4(0) - 2 + k\{5(0) - 7(0) + 3\} = 0$

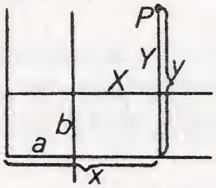
or, $3k = 2$. Substituting this value into the equation gives

$(3x + 4y - 2) + \frac{2}{3}(5x - 7y + 3) = 0$

i.e. $19x - 2y = 0$

CHANGE OF ORIGIN

In this section we consider the effect of **changing the axes without rotation.** That is, the new axes are **parallel** to the old.



Let the point P be (x, y) and (X, Y) referred to the old and new axes respectively, and let the new origin be (a, b) referred to the old axes.

Then $\mathbf{x = X + a}$ and $\mathbf{y = Y + b}$.

Hence if an equation referred to the old axes was $f(x, y) = 0$ then it will become $f(X + a, Y + b) = 0$.
It should be noted that the degree of an equation cannot be altered by any change of axes. This is true even for changes with rotation and to oblique axes.

Example Transform the equation $\frac{x}{a} + \frac{y}{b} - 1 = 0$ by referring it to parallel axes through the point $(0, b)$.
Substituting $x = X$, $y = Y + b$ gives

$$\frac{X}{a} + \frac{Y + b}{b} - 1 = 0$$

i.e. $bX + aY = 0$

Similarly, transform the origin to the point $(-3, -2)$ in the equation $x^2 + y^2 + 6x + 4y - 12 = 0$.

Substituting $x = X - 3$, $y = Y - 2$ gives

$$(X - 3)^2 + (Y - 2)^2 + 6(X - 3) + 4(Y - 2) - 12 = 0$$

$$X^2 - 6X + 9 + Y^2 - 4Y + 4 + 6X - 18 + 4Y - 8 - 12 = 0$$

$$X^2 + Y^2 = 25$$

In both of the above, the equations have been reduced to an algebraically simpler form by the change of origin. This device can be used to make working easier but final results must be returned to the original axes.

THE CIRCLE

A circle is the locus of a point (x, y) which moves so that its distance from a fixed point (h, k) is always equal to **a.**
Using Pythagoras's theorem we see that the equation connecting x and y is

$$(\mathbf{x} - \mathbf{h})^2 + (\mathbf{y} - \mathbf{k})^2 = \mathbf{a}^2$$

which is, therefore, the equation of the circle. Expanding,

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - a^2 = 0.$$

Note that the coefficients of x^2 and y^2 are the **same** and that there is **no** term in **xy.**
The general equation of a circle is therefore

$$\mathbf{x}^2 + \mathbf{y}^2 + \mathbf{2gx} + \mathbf{2fy} + \mathbf{c} = \mathbf{0} \qquad (1)$$

Compare this equation with the one above. Given the equation of a circle in general form, the coordinates of the centre are

$$(-\mathbf{g}, -\mathbf{f})$$

and the radius **a** is

$$\sqrt{\mathbf{g}^2 + \mathbf{f}^2 - \mathbf{c}}.$$

In particular, if the **origin is the centre** of the circle $h = k = 0$, and the equation becomes

$$\mathbf{x}^2 + \mathbf{y}^2 = \mathbf{a}^2. \qquad (2)$$

In the example on the previous page we transformed a circle so that its centre was the origin.
It is very convenient to express the coordinates of a point P on the circle

$$x^2 + y^2 = a^2$$

in parametric form, that is, to express x and y in terms of a third variable.

Since $a^2(\cos^2 \theta + \sin^2 \theta) = a^2$

we take $\mathbf{x} = \mathbf{a} \cos \theta$, $\mathbf{y} = \mathbf{a} \sin \theta$.

The equation of the tangent at $(a \cos \theta, a \sin \theta)$ is

$$(y - a \sin \theta) = m(x - a \cos \theta)$$

where m is the value of $\frac{dy}{dx}$ at $(a \cos \theta, a \sin \theta)$

Since $\dfrac{dx}{d\theta} = -a \sin\theta$ and $\dfrac{dy}{d\theta} = a\cos\theta$, then

$$\frac{dy}{dx} = \frac{dy}{d\theta}\bigg/\frac{dx}{d\theta} = -\frac{\cos\theta}{\sin\theta}$$

Substituting, the equation of the tangent is

$$\mathbf{x \cos\theta + y \sin\theta = a.} \qquad (4)$$

If (x_1, y_1) is the point $(a\cos\theta, a\sin\theta)$, equation (4) can be written

$x(a\cos\theta) + y(a\sin\theta) = a^2.$

$$\mathbf{xx_1 + yy_1 = a^2} \qquad (5)$$

Similarly, the equation of the tangent at (x_1, y_1) to the circle given by

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

is $\mathbf{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0}$ (6)

Note. The relation between (6) and (1) should be looked at carefully because it illustrates a general rule for finding the tangent to a curve at the point (x_1, y_1) on the curve.

Squared terms become xx_1 or yy_1, linear terms, say $2x$, become $x + x_1$. Applying this to the parabola $y^2 = 4ax$ gives the tangent $yy_1 = 2a(x + x_1)$.

LENGTH OF TANGENT

If t is the length of the tangent from (x_1, y_1) to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0,$

then $\mathbf{t^2 = x_1{}^2 + y_1{}^2 + 2gx_1 + 2fy_1 + c}$ (1)

This follows easily from the distance formula.

Notes. 1. Before this expression can be used, the equation of the circle must be written with the **coefficients** of x^2 and y^2 equal to **unity.** (See example.)

2. If the expression (1) has a **negative** value, the point (x_1, y_1) is **inside** the circle.

Example Find the length of the tangent to the circle $3x^2 + 3y^2 + 6x - 4 = 0$ from the point (4, 2).

Writing $x = 4$ and $y = 2$ in the equation (1) gives

$t^2 = 16 + 4 + 8 - \frac{4}{3} = 26\frac{2}{3} \quad \therefore\ t = \sqrt{26\frac{2}{3}}$

THE NORMAL AT (x_1, y_1)

The normal at any point on a curve is the straight line through that point perpendicular to the tangent.
Clearly in the case of the circle all normals pass through the centre.
For the general circle we require the line through $(-g, -f)$ and (x_1, y_1).

$\therefore$ the normal is $\dfrac{y - y_1}{y_1 + f} = \dfrac{x - x_1}{x_1 + g}$ (see page 66)

CIRCLE ON (x_1, y_1), (x_2, y_2) AS DIAMETER

If A, B are the points (x_1, y_1), (x_2, y_2) and $P(x, y)$ is any point on the circumference of the required circle, $\angle APB = 90°$. Hence AP is perpendicular to PB or

$$\frac{y - y_1}{x - x_1} \cdot \frac{y - y_2}{x - x_2} = -1 \qquad (m_1 m_2 = -1)$$

i.e. $\mathbf{(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0}$

Circle through three points

Let the points be (x_1, y_1), (x_2, y_2), (x_3, y_3) and suppose the required circle is denoted by $x^2 + y^2 + 2gx + 2fy + c = 0$. Then since (x_1, y_1) lies on the circle

$$x_1{}^2 + y_1{}^2 + 2gx_1 + 2fy_1 + c = 0,$$

and two similar equations hold for (x_2, y_2), (x_3, y_3).
These three equations are sufficient to determine the values of g, f and c.

Alternative method

In numerical problems it is often easy to write down the equations of the **perpendicular bisectors of two chords** of the circle.
Solving these equations determines the coordinates of the centre of the circle, whose radius can then be calculated.

Distance quadratic

If through a fixed point $P(h, k)$ a line is drawn making an angle θ with the x-axis, then if Q is a point such that $PQ = r$ units, Q is the point

whose coordinates are

$x = h + r\cos\theta$

$y = k + r\sin\theta$

If r is allowed to vary, this gives the current coordinates of any point in the line.
If this line meets the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ at A and B,

PA and PB are the roots of the quadratic equation

$$\mathbf{(h + r\cos\theta)^2 + (k + r\sin\theta)^2 + 2g(h + r\cos\theta)}$$
$$\mathbf{+ 2f(k + r\sin\theta) + c = 0}$$

INTERSECTION OF LINE AND CIRCLE

Given the equations of a straight line (L) and a circle (S), solving the pair as simultaneous equations gives us the coordinates of points satisfying both equations, i.e. the coordinates of the points of intersection.
If we should require the equation of another circle through these points then it is not necessary to solve the equations but simply to write

$$\mathbf{S + kL = 0}$$

which gives the whole family of circles through the points according to the value of k (see also below and page 69).

COMMON CHORDS AND $S_1 + kS_2 = 0$

Any circle through the intersections of

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \qquad (1)$$

$$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \qquad (2)$$

is given by an equation of the form

$$\mathbf{x^2 + y^2 + 2g_1x + 2f_1y + c_1 + k(x^2 + y^2 + 2g_2x + 2f_2y + c_2) = 0} \qquad (3)$$

This follows (a) because it is the **equation of a circle**—coefficients of x^2 and y^2 are equal, and there is no term in xy; (b) points whose coordinates satisfy (1) and (2) **also satisfy (3) for any** value of **k;** (c) any circle through the intersections of (1) and (2) is fixed if we give the coordinates of a third point on it. The coordinates of this point

substituted in (3) give one and only one value for **k.**
The equation of such a circle can therefore be written down **without finding the co-ordinates** of the **intersections** of the circle. If $k = -1$, equation (3) reduces to a first degree equation. Since it **must** pass through the intersections of the circles, it is the **equation of the common chord.**

CHORD JOINING TWO POINTS

If P is $(a \cos \theta, a \sin \theta)$ and Q is $(a \cos \phi, a \sin \phi)$ the gradient of PQ is

$$\frac{a \sin \theta - a \sin \phi}{a \cos \theta - a \cos \phi}$$

$$= \frac{a.2 \cos \tfrac{1}{2}(\theta + \phi) \sin \tfrac{1}{2}(\theta - \phi)}{-a.2 \sin \tfrac{1}{2}(\theta + \phi) \sin \tfrac{1}{2}(\theta - \phi)}$$

$$= - \cot \tfrac{1}{2}(\theta + \phi)$$

Hence the chord PQ is

$$y + x. \frac{\cos\frac{1}{2}(\theta + \phi)}{\sin \frac{1}{2}(\theta + \phi)} = a \sin \theta + \frac{a \cos \theta \cos \frac{1}{2}(\theta + \phi)}{\sin \frac{1}{2}(\theta + \phi)}$$

$$\text{or } x \cos \frac{\theta + \phi}{2} + y \sin \frac{\theta + \phi}{2} = a \cos \frac{\theta - \phi}{2}.$$

LINE TOUCHING A CIRCLE

Consider the circle $\mathbf{x^2 + y^2 = a^2}$ and the line $\mathbf{y = mx + c}$
They intersect where

$$x^2 + (mx + c)^2 = a^2$$

i.e. $x^2(1 + m^2) + 2mcx + c^2 - a^2 = 0$

This equation offers three possible solutions.

(i) Two real roots. The line cuts the circle.
(ii) Imaginary roots. The line misses the circle completely.
(iii) Equal roots. The line touches the circle. i.e. it is a tangent. This situation arises when '$b^2 = 4ac$' (see page 11).

i.e. $4m^2c^2 = 4(1 + m^2)(c^2 - a^2)$

i.e. $c^2 = a^2(1 + m^2) \qquad c = \pm a\sqrt{1 + m^2}$

Therefore, for all values of m, the lines

$\mathbf{y = mx \pm a\sqrt{1 + m^2}}$

will be tangents to the circle $x^2 + y^2 = a^2$.

THE PARABOLA

The locus of a point which moves so that its **distance from a fixed line** called the **Directrix** is always **equal** to its **distance from a fixed point** called the **Focus** is a **Parabola.**
The axes are chosen so that the **focus** S has coordinates **(a, O)** and the **directrix** has the equation $\mathbf{x + a = O}$.
The equation of the parabola is then $\mathbf{y^2 = 4ax}$.
The parabola is symmetrical about the x-axis, a fact that can prove very useful in solving problems.
The coordinates of a point $P(x, y)$ on the parabola can be expressed in terms of a parameter t.

$\mathbf{x = at^2,\ y = 2at.}$

The **tangent** at the point $(at^2, 2at)$ is

$\mathbf{ty = x + at^2,}$

and the **normal** is

$\mathbf{tx + y = at^3 + 2at.}$

For the gradient of the tangent we have

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$
$$= 2a \div 2at$$
$$= \frac{1}{t}$$

CHORD JOINING TWO POINTS

The gradient of the chord joining the points $(at_1{}^2, 2at_1)$ and $(at_2{}^2, 2at_2)$ is

$$\frac{2at_1 - 2at_2}{at_1{}^2 - at_2{}^2}$$

$$= \frac{2a(t_1 - t_2)}{a(t_1 - t_2)(t_1 + t_2)}$$

$$= \frac{2}{t_1 + t_2}$$

$\therefore$ the chord has an equation of the form

$(t_1 + t_2)y - 2x = k.$

But the point $(at_1{}^2, 2at_1)$ lies on the chord, whose equation is thus found to be $\mathbf{(t_1 + t_2)y - 2x = 2at_1t_2}$.
A chord through the focus, **perpendicular** to the axis is called the **latus rectum.** Its total length is $4a$, and its equation is $x = a$.

GEOMETRICAL PROPERTIES

The tangent $ty = x + at^2$ cuts the y-axis at $(0, at)$, and the x-axis at $(-at^2, 0)$.
Hence if the tangent at **P** meets the y-axis at **T** and the x-axis at **Q**, and the ordinate at **P** meets the x-axis at **M,**

PM = 2OT and **QO = OM**

See also the example at the end of this section.

FOCAL CHORDS

If the point $(a, 0)$ lies on the chord joining 't_1', 't_2'

i.e. $(t_1 + t_2)y - 2x = 2at_1t_2$,

then $(t_1 + t_2).0 - 2.a = 2at_1t_2$

or $\mathbf{t_1t_2 = -1}$. (1)

Hence if P, Q are the extremities of a chord passing through the focus, the **parameters** of P, Q are **related by equation (1).**

INTERSECTION OF TANGENTS

The tangents at any two points $P(at_1{}^2, 2at_1)$ and $Q(at_2{}^2, 2at_2)$ are given by

$t_1y - x = at_1{}^2$

$t_2y - x = at_2{}^2$

Solving these, we have $(t_1 - t_2)y = a(t_1{}^2 - t_2{}^2)$

or $y = a(t_1 + t_2)$,

and substitution gives $x = at_1t_2$.

Hence the tangents intersect at $\mathbf{[at_1t_2, a(t_1 + t_2)]}$

FOCUS—DIRECTRIX PROPERTY

If PQ passes through the focus, $t_1t_2 = -1$, so that the **tangents** at the **ends of this focal chord** intersect at

$[-a, a(t_1 + t_2)]$

i.e. **intersect on the line x = −a, the directrix.**
Further, since the gradients of the tangents are

$\frac{1}{t_1}$ and $\frac{1}{t_2}$, we have $\frac{1}{t_1} \cdot \frac{1}{t_2} = -1$,

so that the **tangents at the ends of a focal chord intersect at right angles on the directrix.**

LINE TOUCHING A PARABOLA

Since the gradient of the tangent at t is $\frac{1}{t}$, we have, writing $m = \frac{1}{t}$ in the equation $ty = x + at^2$,

$$\frac{y}{m} = x + \frac{a}{m^2}$$

i.e. $\mathbf{y = mx + \frac{a}{m}}$

This means that any line with equation of this form is a tangent to the parabola $y^2 = 4ax$.
The result could have been derived in the same way as that for a circle. (See pages 75, 76)

Example By using their gradients show that the focal chord and a line drawn parallel to the axis from the point $P(x_1, y_1)$ on the parabola $y^2 = 4ax$ are equally inclined to the normal at P.
Let the focal chord be PS, the normal PN and the parallel line be PL

Then gradient of $PS = \frac{y-O}{x-a} = \frac{y}{x-a} = m_1$

gradient of $PN = \dfrac{-y_1}{2a} = m_2$

gradient of $PL = 0 = m_3$

Using the formula for the angle between two lines (page 67) we obtain

$$S\hat{P}N = \tan^{-1}\left(\frac{\dfrac{y_1}{x_1 - a} + \dfrac{y_1}{2a}}{1 - \dfrac{y_1}{2a} \cdot \dfrac{y_1}{x_1 - a}}\right)$$

Multiply both numerator and denominator by $2a(x_1 - a)$.

$$= \tan^{-1}\left(\frac{ay_1 + x_1y_1}{2ax_1 - 2a^2 - y_1^{\,2}}\right)$$

Substitute $y^2{}_1 = 4ax_1$ as the point is on the parabola.

$$= \tan^{-1}\left(\frac{y_1(a + x_1)}{-2a(a + x_1)}\right) = \tan^{-1}\left(\frac{-y_1}{2a}\right)$$

$$\text{Similarly } L\hat{P}N = \tan^{-1}\left(\frac{\dfrac{-y_1}{2a} - 0}{1 - 0}\right) = \tan^{-1}\left(\frac{-y_1}{2a}\right)$$

Therefore the two lines are equally inclined to the normal. This is an exceedingly important practical property as it means that a ray of light leaving the focus of a parabolic mirror is reflected parallel to the axis. This is used in car headlamps, searchlights, radar aerials etc. The converse is also true and is fundamental to radio telescope design.

THE ELLIPSE

The locus of a point, which moves so that its **distance from a fixed point** called the **Focus** divided by its **distance from a fixed line** called the **Directrix** is equal to a constant e $(e < 1)$, is an **Ellipse.**

The constant e is called the **Eccentricity.**

The axes are chosen so that the **focus** has coordinates **(ae, O)** and the **directrix** has the equation

$$\mathbf{x} = \frac{\mathbf{a}}{\mathbf{e}}$$

The equation of the ellipse is then

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

For convenience write $\mathbf{b^2 = a^2(1 - e^2)}$ and the equation becomes

$$\frac{\mathbf{x}^2}{\mathbf{a}^2} + \frac{\mathbf{y}^2}{\mathbf{b}^2} = \mathbf{1} \qquad (1)$$

The ellipse cuts the x-axis at $(a, 0)$ and $(-a, 0)$ and so has a **Major** axis of length **2a.**
It cuts the y-axis at $(0, b)$ and $(0, -b)$ and so has a **Minor axis** of length **2b.**
Since the ellipse is symmetrical about the two axes, there is a second directrix whose equation is

$x = -\dfrac{a}{e}$ and a second focus at $(-ae, o)$

The parametric coordinates of a point on the ellipse are

$$\mathbf{x = a \cos \theta, \qquad y = b \sin \theta} \qquad (2)$$

The tangent at this point is

$$\frac{\mathbf{x}}{\mathbf{a}} \mathbf{\cos \theta} + \frac{\mathbf{y}}{\mathbf{b}} \mathbf{\sin \theta = 1} \qquad (3)$$

Here $\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \div \dfrac{dx}{d\theta} = \dfrac{b \cos \theta}{-a \sin \theta}$

Hence the tangent is $y - b \sin \theta = -\dfrac{b \cos \theta}{a \sin \theta}(x - a \cos \theta)$

or $\dfrac{y \sin \theta}{b} - \sin^2 \theta = -\dfrac{x \cos \theta}{a} + \cos^2 \theta$

from which equation (3) follows.
Note that
(i) a parabola is a special case of an ellipse with $e = 1$,
(ii) a circle is a special case of an ellipse with $e = 0$.
The area of a circle, πr^2, is a special case of the area of the ellipse, πab.

TANGENT AND NORMAL

The tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is

$$\mathbf{\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1}$$

Note the similarity with the corresponding result for the circle. The normal at $(a \cos \theta, b \sin \theta)$ for the ellipse

is $\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2$ (4)

Notice (1) that **inverting** the **coefficients** and **altering one sign** in the L.H.S. of the tangent equation (3) on page 80 produces a **perpendicular** line, and (2) that the R.H.S. of equation (4) expresses the fact that this perpendicular to the tangent passes through $(a \cos \theta, b \sin \theta)$.

Focal distances

If P is a point on the ellipse, S, S' are the foci, and PM, PM' are the perpendiculars from P to the two directrices,

$SP = e.PM$ and $S'P = e.PM'$

Therefore $\mathbf{SP + S'P} = e(PM + PM')$

$$= e \cdot \frac{2a}{e}$$

$$= \mathbf{2a}.$$

Hence the **sum of the focal distances** for any point on the ellipse is **equal to the length of the major axis.**

Relation between a, b, e

If B is an end of the minor axis, the sides of the right-angled triangle OSB are b, ae and a. This provides an easy way of remembering the relationship

$$\mathbf{b^2 = a^2 - a^2e^2}.$$

LINE TOUCHING AN ELLIPSE

As on pages 75 and 76 we can find the equation of a line such that it touches the ellipse. The two lines of gradient m which touch the ellipse are

$$\mathbf{y = mx \pm \sqrt{a^2m^2 + b^2}} \tag{5}$$

DIRECTOR CIRCLE

The tangents perpendicular to (5) are

$$y = -\frac{1}{m}x \pm \sqrt{\frac{a^2}{m^2} + b^2}$$

or $my + x = \pm\sqrt{a^2 + b^2m^2}$ (6)

(5) can be written $y - mx = \pm\sqrt{a^2m^2 + b^2}$ (7)

If we eliminate m between equations (6) and (7) we obtain the **locus of the intersection of perpendicular tangents** to the ellipse. Squaring and adding, $\mathbf{x^2 + y^2 = a^2 + b^2}$ is the required locus.
Notice that this is the equation of a circle which is concentric with the ellipse, and the square of its radius is equal to the sum of the squares of the semi-axes. This circle is called the **director circle** of the ellipse.

Example Two tangents are drawn from the point $R(x_1, y_1)$ to the ellipse $x^2/a^2 + y^2/b^2 = 1$ and touch it at the points $P(a\cos\alpha, b\sin\alpha)$ and $Q(a\cos\beta, b\sin\beta)$. Show that

$$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{ay_1}{bx_1}$$

Let the tangent from R to P be $\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1$

This passes through (x_1, y_1) so that

$$\frac{x_1}{a}\cos\alpha + \frac{y_1}{b}\sin\alpha = 1 \tag{1}$$

Similarly for tangent RQ

$$\frac{x_1}{a}\cos\beta + \frac{y_1}{b}\sin\beta = 1 \tag{2}$$

Subtract (2) from (1)

$$\frac{-2x_1}{a}\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)+\frac{2y_1}{b}\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)=0$$

hence $\tan\left(\frac{\alpha+\beta}{2}\right)=\frac{ay_1}{bx_1}$

THE HYPERBOLA

The locus of a point, which moves so that its **distance from a fixed point** called the **Focus** divided by its **distance from a fixed line** called the **Directrix** is equal to a constant $e(e>1)$, is a **Hyperbola.**

The axes are chosen as for the ellipse. The equation of the hyperbola is then

$$\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1$$

or, writing $b^2=a^2(e^2-1)$

$$\mathbf{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.}$$

The two **foci** are **($\pm$ae, O)** and the two **directrices** are

$$\mathbf{x=\pm\frac{a}{e}}$$

The **parametric co-ordinates** of a point on the hyperbola are **x = a sec θ,** **y = b tan θ.**

Note, an alternative form of parametric coordinates is $x=a\cosh\theta$, $y=b\sinh\theta$.

The equation of the tangent at $(a\sec\theta, b\tan\theta)$ is

$$\mathbf{\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1.}$$

The line $\mathbf{y=\frac{b}{a}x}$ cuts the hyperbola where

$$\frac{x^2}{a^2}-\frac{x^2}{a^2}=1$$

i.e. $x^2 . 0 = 1$

It therefore cuts the hyperbola at $x = \infty$ and $x = -\infty$ and so is a **tangent at infinity.** Such a line is called an **Asymptote.** Similarly the line

$$\mathbf{y} = -\frac{\mathbf{b}}{\mathbf{a}}\mathbf{x} \qquad \text{is an asymptote}$$

If $b = a$, the two asymptotes are $y = x$ and $y = -x$. These two lines are **perpendicular** to each other and then the hyperbola is called a **Rectangular Hyperbola.** Its equation is

$$x^2 - y^2 = a^2$$

It is more convenient to **rotate** the axes through 45°, so that the **asymptotes become the axes.** This equation then becomes

$$\mathbf{xy} = \mathbf{c}^2.$$

RECTANGULAR HYPERBOLA $xy = c^2$

The parametric coordinates of a point on $xy = c^2$ are

$$\mathbf{x} = \mathbf{ct}, \ \mathbf{y} = \frac{\mathbf{c}}{\mathbf{t}} \tag{1}$$

Since $\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt} = -\dfrac{c}{t^2} \div c = -\dfrac{1}{t^2}$

the **tangent** at $\left(ct, \dfrac{c}{t}\right)$ is

$$\mathbf{x} + \mathbf{yt}^2 = \mathbf{2ct} \tag{2}$$

and the **normal** is

$$\mathbf{xt}^2 - \mathbf{y} = \mathbf{ct}^3 - \frac{\mathbf{c}}{\mathbf{t}} \tag{3}$$

Notice again the relationship between equations (2) and (3).

THE CHORD JOINING TWO POINTS P, Q

The gradient of the line joining $\left(ct_1, \frac{c}{t_1}\right)$ and $\left(ct_2, \frac{c}{t_2}\right)$

$$= \left(\frac{c}{t_1} - \frac{c}{t_2}\right) \div (ct_1 - ct_2)$$

$$= \frac{c(t_2 - t_1)}{t_1 t_2} \times \frac{1}{c(t_1 - t_2)}$$

$$= \frac{-1}{t_1 t_2}$$

$\therefore$ the required chord has an equation of the form

$t_1 t_2 y + x = k,$

where k is chosen so that $\left(ct_1, \frac{c}{t_1}\right)$ lies on this line.

The result is the equation symmetric in t_1, t_2,

$$\mathbf{t_1 t_2 y + x = c(t_1 + t_2)} \tag{8}$$

If PQ produced cuts the x-axis at A and the y-axis at B

A is the point $[c(t_1 + t_2), 0]$,

while B is the point $\left[0, \frac{c}{t_1} + \frac{c}{t_2}\right]$

The mid-points of AB and PQ are easily seen to be the same, which means that **PA** and **QB** are equal.

Similarly if the tangent at **R,** the point $\left(ct, \frac{c}{t}\right)$ cuts the axes at C and D,

R is the mid-point of CD.

Example Find the area of the triangle formed by the axes of x and y and the tangent to the hyperbola $xy = c^2$ at $P(ct, c/t)$.
The tangent has equation $x + t^2 y = 2ct$

It meets the x-axis at $x = 2ct$. i.e. the point $Q(2ct, 0)$

It meets the y-axis at $y = 2c/t$. i.e. the point $R(0, 2c/t)$

Area of triangle $= \frac{1}{2} \cdot OQ \cdot OR = \frac{1}{2} \cdot 2ct \cdot 2c/t$
$= 2c^2$

Algebra of Complex Numbers

Define the symbol **i** by $\mathbf{i}^2 = -1$ or $\mathbf{i} = \sqrt{-1}$.
Then, if **a** and **b** are **real numbers,** any number of the form $\mathbf{a + ib}$ is called a **Complex Number**; **a** is called the **Real part** and **b** the **imaginary** part.
Taking rectangular cartesian axes, **the complex number x + iy** can be **represented uniquely** by the point **(x, y).** Such a diagram is called an **Argand Diagram.**
The **sum** and **difference** of $x_1 + iy_1$ and $x_2 + iy_2$ are defined as $\mathbf{(x_1 \pm x_2) + i(y_1 \pm y_2)}$

If P_1 is the point (x_1, y_1), P_2 is the point (x_2, y_2) and R is the point $(x_1 + x_2, y_1 + y_2)$ on the Argand diagram, it is easy to see that **P_1, P_2, R and the origin O form a parallelogram.** It follows, therefore, that complex numbers follow the **Vector Law of Addition.**
Remember, a **complex number** is a **Vector.**
The **multiplication** of complex numbers follows the **ordinary laws of algebra.**

$$\begin{aligned} \text{e.g. } (2 + 3i) \times (4 - 5i) &= 8 + 12i - 10i - 15i^2 \\ &= 8 + 2i + 15 \\ &= 23 + 2i \end{aligned}$$

Note in particular, that

$$x^2 + y^2 = (x + iy)(x - iy)$$

The numbers $x_1 + iy_1$ and $x_1 - iy_1$ are called **Conjugate numbers** and their product is always **real** and **positive.**

$\sqrt{x_1{}^2 + y_1{}^2}$ is called the **Modulus** of $x_1 + iy_1$ and is written $|x_1 + iy_1|$. The modulus is the **distance from the origin** to the representation of $x_1 + iy_1$ on the Argand diagram.
The division of two complex numbers is carried out by multiplying the numerator and the denominator by the **conjugate of the denominator.**

$$\text{e.g. } \frac{2 + 3i}{4 - 5i} = \frac{(2 + 3i) \times (4 + 5i)}{(4 - 5i) \times (4 + 5i)} = \frac{-7 + 22i}{41} = \frac{-7}{41} + \frac{22}{41}i.$$

Complex numbers arise first in **solving quadratic equations** for which $\mathbf{b^2 - 4ac < 0}$.

For example, the equation $x^2 - 6x + 25 = 0$ gives

$$x = \frac{6 \pm \sqrt{36 - 100}}{2} = \frac{6 \pm 8\sqrt{-1}}{2} = 3 \pm 4i$$

Note (1) that the solutions of a quadratic equation with real coefficients and complex roots will always be a pair of conjugate numbers; (2) the sum of the complex roots is real $(-b/a)$ and (3) the product of them is also real (c/a).

Notice the similarities in dealing with **conjugate surds** and **conjugate complex numbers.**

Surds

$(7 + \sqrt{3}) + (7 - \sqrt{3}) = 14$ (rational)

$(7 + \sqrt{3})(7 - \sqrt{3}) = 49 - 3 = 46$ (rational)

$$\frac{1}{7 + \sqrt{3}} = \frac{7 - \sqrt{3}}{(7 + \sqrt{3})(7 - \sqrt{3})} = \frac{7 - \sqrt{3}}{46} \text{ (rational denominator)}$$

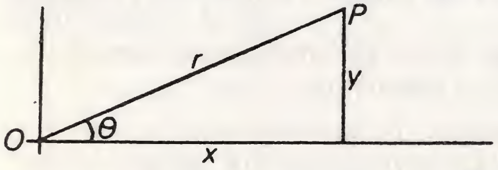
Complex numbers

$(7 + 3i) + (7 - 3i) = 14$ (real)

$(7 + 3i)(7 - 3i) = 49 + 9 = 58$ (real)

$$\frac{1}{7 + 3i} = \frac{7 - 3i}{(7 + 3i)(7 - 3i)} = \frac{7 - 3i}{58} \quad \text{(real denominator)}$$

POLAR FORM



Consider the point $P(x, y)$ in an Argand diagram. If $OP = r$ and OP makes an angle θ with the positive x-axis then we can write $x = r \cos \theta$ and $y = r \sin \theta$.

Thus $\mathbf{x + iy = r(\cos\theta + i \sin\theta)}$. This is the polar form of the complex number.

In dealing with complex numbers, the function $\cos \theta + i \sin \theta$ is of fundamental importance.

We have $(\cos \theta + i \sin \theta)(\cos \theta - i \sin \theta) = \cos^2 \theta + \sin^2 \theta = 1$

so that $\dfrac{\mathbf{1}}{\mathbf{cos}\ \theta + \mathbf{i\ sin}\ \theta} = \mathbf{cos}\ \theta - \mathbf{i\ sin}\ \theta.$

Note that $r = \sqrt{x^2 + y^2} = |x + iy|$.

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_2)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

Then $\mathbf{z}_1\mathbf{z}_2 = r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$$= r_1r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)\}$$

$$= \mathbf{r_1r_2(cos(\theta_1 + \theta_2) + i\ sin(\theta_1 + \theta_2))}$$

This result generalizes to

$$z_1z_2z_3\ldots z_n = r_1r_2r_3\ldots r_n(\cos(\theta_1 + \theta_2 + \theta_3 + \cdots + \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + \cdots + \theta_n))$$

In particular, when $z_1 = z_2 = z_3 \ldots = z_n$

$$\mathbf{z^n = r^n(cos\ n\theta + i\ sin\ n\theta)}$$

If also $r = 1$ then $z = \cos\theta + i\sin\theta$. The last result then becomes $z^n = \mathbf{(cos\ \theta + i\ sin\ \theta)^n = cos\ n\theta + i\ sin\ n\theta}$

This result is known as **de Moivre's Theorem.** It is also true for n negative.

THE CUBE ROOTS OF UNITY

Consider the equation $z^n = 1$, where n is a positive integer. A solution of this equation is called the **nth root of unity**. The only real solutions possible are $z = 1$ if n is odd and $z = \pm 1$ if n is even. If complex roots are allowed then they are given by

$$\mathbf{z = cos\ \frac{2k\pi}{n} + i\ sin\ \frac{2k\pi}{n}} \qquad k = 0, 1, 2, \ldots (n - 1)$$

In particular if $n = 3$ then the **cube roots of unity** are

$z_0 = \cos 0 + i\sin 0 = 1$ by putting $k = 0$

$z_1 = \cos 2\pi/3 + i\sin 2\pi/3 = -\frac{1}{2} + i\sqrt{3}/2$ by putting $k = 1$

$z_2 = \cos 4\pi/3 + i\sin 4\pi/3 = -\frac{1}{2} - i\sqrt{3}/2$ by putting $k = 2$

Notice that using further values of k does not produce more solutions but only repeats of the three so far obtained.

Notice also that the complex roots are conjugate.

EXPONENTIAL FORM

Let $x = i\theta$. Then $e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \cdots$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

The real and imaginary parts of this series are the Maclaurin expansions of $\cos\theta$ and $\sin\theta$. (See page 54)

$\therefore$ $\mathbf{e^{i\theta} = \cos\theta + i \sin\theta}$

By this means we can write the complex number $r(\cos\theta + i\sin\theta)$ as $re^{i\theta}$, and by writing $-\theta$ for θ obtain $r(\cos\theta - i\sin\theta) = re^{-i\theta}$

Example Prove that $\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$. Hence find all the solutions of the equation $16\sin^5\theta = \sin 5\theta$.

This result may be proved by application of the many formulae given on page 32 but de Moivre's theorem offers a simpler way.

$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$

$$= c^5 + 5c^4 is - 10c^3s^2 - 10c^2 is^3 + 5cs^4 + is^5$$

remembering that $i^2 = -1$, $i^3 = -i$ and where $c = \cos\theta$, $s = \sin\theta$. Equate imaginary parts of the identity,

$$\begin{aligned} \mathbf{\sin 5\theta} &= 5c^4s - 10c^2s^3 + s^5 \\ &= 5(1-s^2)^2s - 10(1-s^2)s^3 + s^5 \\ &= 5s - 10s^3 + 5s^5 - 10s^3 + 10s^5 + s^5 \\ &= 5s - 20s^3 + 16s^5 \\ &= \mathbf{5\sin\theta - 20\sin^3\theta + 16\sin^5\theta} \end{aligned}$$

If $16\sin^5\theta = \sin 5\theta$

$$0 = 5\sin\theta - 20\sin^3\theta = 5\sin\theta(1 - 4\sin^2\theta)$$

$$\therefore \quad \sin\theta = 0, \pm\tfrac{1}{2}$$

$$\begin{aligned} \theta &= k\pi,\ k\pi \pm \pi/6 \\ &= k\pi,\ (6k \pm 1)\pi/6 \end{aligned}$$

LINEAR LAW

If two variables **x** and **y** are connected by some law, experimental data can be obtained showing corresponding values of x and y. The values can be plotted against each other to show the law visually as a curve. However, **the only curve which can be recognized with certainty is the straight line.**

To obtain a straight line, the **plotted** variables, say **X** and **Y,** must be connected by a relationship of the form **Y = mX + c,** and this can be done by a **suitable choice** of X and Y.

e.g. $y^3 = mx^2 + c$ will **not** give a straight line, but the substitution $y^3 = Y$ and $x^2 = X$ gives the line **Y = mX + C**, from which **m** and **c** can be deduced.

The kind of relationship between x and y is always given and all that remains is the choice of X and Y,

e.g. $y^2 = ax^2 + bx$.

$$\therefore \quad \frac{y^2}{x} = ax + b.$$

Writing $Y = \dfrac{y^2}{x}$ and $X = x$ we get

$\mathbf{Y = aX + b}$

Hence, plotting $\dfrac{\mathbf{y}^2}{\mathbf{x}}$ against **x** will produce a **straight line.**

e.g. $\quad y = ae^{bx}$

$\therefore \log y = \log a + bx.$

Hence plot **log y** against **x.**

Do **not** attempt to find **a** and **b** by substituting **given** values. Some of them may be inaccurate. The given values should be plotted as described and the **best straight line drawn through them.** The **gradient** can now be read off, paying careful attention to the **scales** involved.

The constant **c** in the $Y = mX + c$ equation can be read off **from the graph** if the graph cuts the line $X = O$.

Otherwise **m** and **c** may be found from reading off the coordinates of **two points on the graph** (preferably two with simple X values).

Given the table of values

x	1	3	5	8
y	3	4·6	5·1	3·9

it is thought that x and y are connected by an equation of the type

$y^2 = ax^2 + bx$

In the case $\dfrac{y^2}{x} = ax + b$, and we complete the table

$x = X$	1	3	5	8
$\dfrac{y^2}{x} = Y$	9	7·05	5·2	1·90

If the graph of this function is drawn, it will be seen (a) that the **points are nearly collinear** and (b) that equation of the straight line appears to be

$\mathbf{Y = 10 - X,}$

so that the (x, y) relationship is

$$\frac{y^2}{x} = 10 - x,$$

or $\mathbf{y^2 = 10x - x^2}$

Again if we have the table,

x	0·5	1	3	5
y	4·26	6·02	24	97

with a suspected relationship $y = a\,.\,b^x$

we have $\log y = \log a + x \log b$,
and make up the table,

$x = X$	0·5	1	3	5
$\log y = Y$	0·6294	0·7796	1·3802	1·9868

Rounding off the Y values suitably, we have

(a) the points are nearly collinear;

(b) when $X = 0$, $\log y \simeq 0{\cdot}48$,
i.e. $\mathbf{\log a \simeq 0{\cdot}48}$
or $a = 3$

(c) the gradient of the straight line is approximately $0{\cdot}30$, so that $\mathbf{\log b \simeq 0{\cdot}30}$ and $b = 2$

Hence a relationship $\mathbf{y = 3.2^x}$ appears correct.

CURVES

In the drawing of curves, the following points should be noted.

(i) Find where the curve cuts the **x-axis** by substituting $\mathbf{y = 0}$.

(ii) Find where it cuts the **y-axis** by substituting $\mathbf{x = 0}$.

(iii) **Check for symmetry.** If replacing x by $-x$ does **not alter** the equation, then the curve is **symmetrical about the y-axis.** If replacing y by $-y$ does **not alter** the equation, then the curve is **symmetrical about the x-axis.**

(iv) Check for **asymptotes** in the simple cases when $x \to \infty$ and $y \to \infty$.

(v) Check for **turning points**—Maxima, Minima and Inflexions,

(vi) Check for **limitations of values.** An example of this was seen in the case

$$y = \frac{x^2 + 3x + 3}{x^2 + 2x - 3} \text{ (see page 11)} \qquad (1)$$

when it was shown that y could not lie between certain values. As a further example, given $y^2 = 4 - x$, then clearly $x \leq 4$ for y to exist.

(vii) Find where the curve **cuts** its **horizontal asymptotes.**

(viii) Consider the way in which the curve **approaches its vertical asymptotes.**

$$\text{e.g. } y = \frac{x^2 + 3x + 2}{x^2 + 2x - 3}$$

$$= \frac{(x + 1)(x + 2)}{(x + 3)(x - 1)}$$

We have the following skeleton table.

x	$-\infty$		-5*	-3		-2		-1		0		1		$+\infty$
y	$+1$	$1-$	1	$+\infty$	$-$	0	$+$	0	$-$	$-\frac{2}{3}$	$-$	∞	$+$	1

* The curve **cuts** the line $\mathbf{y = 1}$ at the point where $\mathbf{x = -5}$, and now we have a clear idea of the shape.

The graph $\mathbf{y^2 = f(x)}$

If we sketch first the graph of $Y = f(x)$, and then take **positive and negative square roots of positive ordinates** we obtain the graph of $y^2 = f(x)$.

For example, the circle $y^2 = 9 - x^2$ is obtained in this way from the parabola $Y = 9 - x^2$.

SOLUTION OF EQUATIONS

The solution of various **standard types** is considered first.

(a) Equations of the type $\mathbf{f(x) = 0}$, where $f(x)$ is a polynomial in x. Solutions by the **Remainder Theorem** (page 14) sometimes enable the degree of the equation to be reduced by removing a factor of the form $\mathbf{ax + b}$.

(b) **Simultaneous equations**. If one of these is **linear**, substitute for y or x from the linear equation into the non-linear one, thus reducing it to form (a).

(c) **Equations involving surds.** These equations require care **after** the solutions have been obtained. Remember that the $\sqrt{\ }$ sign means the **positive** square root.

Squaring is an irreversible process.

$x = 3 \Rightarrow x^2 = 9$, but $x^2 = 9 \Rightarrow x = +3$ or $x = -3$.

The **squaring** necessary to get rid of surd terms usually **introduces extraneous roots** as in the following example.

Solve $x + \sqrt{5x + 1} = 7$ (1)

$\therefore \qquad \sqrt{5x + 1} = 7 - x$ (isolating the surd)

$\qquad\qquad 5x + 1 = 49 - 14x + x^2$ (squaring both sides)

$\therefore \quad x^2 - 19x + 48 = 0$ (2)

$\therefore \quad (x - 16)(x - 3) = 0$

$\therefore \quad x = 16$ or $x = 3$

Substitute in original equation:

$16 + \sqrt{81} \neq 7$

$3 + \sqrt{16} = 7$

$\therefore$ **x = 3** is the **only solution.**

Note. $x = 16$ satisfies the equation

$x - \sqrt{5x + 1} = 7,$

which clearly **leads to equation (2)** on eliminating surds.

(d) Equations involving unknowns in the **index.**
It is usually necessary to take logarithms to find a solution.

Solving $5^x = 21$

we have $x \log 5 = \log 21$

$$\text{or } x = \frac{1{\cdot}3222}{0{\cdot}6990} \simeq 1{\cdot}89$$

Note. This division **can** of course **be performed by logs.** Sometimes a substitution can be used, as in the case of the equation $3^{2x+3} - 28.3^x + 1 = 0$, which reduces by putting $Z = 3^x$ to the quadratic form $27Z^2 - 28Z + 1 = 0$.
Very few equations can be solved by formal methods. As it is often very important to find a solution of an equation, an **empirical method** must be used.

To solve the equation f(x) = 0.

(a) **Draw the graph** of $y = f(x)$ and the point x_1, **where it cuts the x-axis,** is a solution of $f(x) = 0$. If this answer is not sufficiently accurate, draw the graph of $y = f(x)$ **over a small range of values** in the neighbourhood of $x = x_1$.

A much enlarged scale can be used to do this and the point where the curve cuts the x-axis can be read off with a **much greater degree of accuracy.**
The process, can, of course, be repeated as often as required.
Note that, if a **very small neighbourhood** of x_1 is used, the graph of $y = f(x)$ over this range will almost be a **straight line** and time can be saved by plotting values at **two points** near x_1, one giving a positive and the other a negative y, and **joining** by a straight line.

(b) **Solve by trial and error.** Used intelligently, this is a quick and effective method. Find x_1 graphically as in (a) and then substitute values of x, near x_1, in $f(x)$ until $f(x)$ becomes as near zero as possible.

(c) **Newton's Method.** Let $f(x) = 0$ have a root **near** to x_1, which may perhaps have been found as in (a). Let the **true root** be $x_1 + h$ so that $f(x_1 + h) = 0$. Apply Taylor's theorem.

Then $f(x_1 + h) = f(x_1) + hf'(x_1) + \dfrac{h^2 f''(x_1)}{2!} + \cdots = 0$

We now make an approximation by ignoring h^2 and higher powers.

$f(x_1) + hf'(x_1) \doteqdot 0$

$$\mathbf{h} \doteqdot \frac{-\mathbf{f}(\mathbf{x}_1)}{\mathbf{f}'(\mathbf{x}_1)}$$

Hence if x_1 is an approximation then, in general, a better value x_2 is given by

$$\mathbf{x}_2 = \mathbf{x}_1 - \frac{\mathbf{f}(\mathbf{x}_1)}{\mathbf{f}'(\mathbf{x}_1)}$$

The process can be repeated indefinitely until a solution of sufficient accuracy is obtained.

Newton's method is an example of an **iterative method** of finding a solution. These methods are of great importance when finding solutions by computer.

Example Let $x_1 = 3$ be an approximate solution of $x^3 + 8x - 46 = 0$

Now $f'(x) = 3x^2 + 8$,

so that $f'(3) = 35$, and $f(3) = 5$

and a **second approximation** is $x = 3 + h_2$, where

$h_2 = -\dfrac{5}{35} \doteqdot -0{\cdot}14$. giving $\mathbf{x}_2 = \mathbf{2{\cdot}86}$.

Now $f(2{\cdot}86) = 0{\cdot}28$ and $f'(2{\cdot}86) = 32{\cdot}54$,

giving $h_3 = -\dfrac{0{\cdot}28}{32{\cdot}54}$ or $\doteqdot -0{\cdot}01$, so that

$\mathbf{x} = \mathbf{2{\cdot}85}$ (to 3 sig, fig.).

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